Simplicial Decompositions of Infinite Graphs

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Chapter 1

Infinite Graphs

Infinite graphs are interesting because the results obtained about them have applications in other branches of mathematics. Many results are applicable to mathematical logic, and it has been shown that a large part of set theory may be regarded as part of the theory of infinite graphs.

Infinite graphs are also interesting in their own right. Frequently, questions about finite graphs, and their answers, have natural extensions to infinite graphs. Conversely, Ramsey theory is an example of a branch of graph theory that originated in formal logic, which may also be regarded as part of infinite graph theory, but has produced many results on finite graphs.

In order to understand the theory of infinite graphs, a familiarity with elementary set theory is required.

1.1 Ordinals and Cardinals

The natural numbers can be defined in terms of finite sets. "0" is taken to be the empty set, $\emptyset$. "1" is the set containing the empty set. "2" is the set containing the empty set, and the set containing the empty set. In summary,

$$\begin{aligned}
0 &= \emptyset \\
1 &= \{\emptyset\} = \{0\} \\
2 &= \{\emptyset, \{\emptyset\}\} = \{0, 1\} \\
3 &= \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} = \{0, 1, 2\} \\
&\vdots \\
n &= \{0, 1, \ldots, n-1\} \\
&\vdots
\end{aligned}$$

By this definition, the natural numbers are known as the *finite ordinals*. Recall that an *order relation* $<$ is irreflexive, asymmetric, and transitive. Note that the finite ordinals are totally ordered by the subset relation $\subset$ and more specifically by the inclusion relation $\in$.

Now this process can be continued past all the natural numbers. The first ordinal larger than all the natural numbers is denoted ω.

$$\begin{aligned} \omega &= \{0, 1, \ldots\} \\ \omega + 1 &= \{0, 1, \ldots, \omega\} \\ &\vdots \end{aligned}$$

Eventually, $\omega + \omega = \omega \cdot 2$ is reached (which is different from $2 \cdot \omega$, but fortunately transfinite arthmetic will not be required in this discussion).

In order to define all the ordinals, we need some more set theory. Let $(A, <)$ be a set with an associated ordering relation. A is *well-ordered* if it is totally ordered and every non-empty subset of A has a smallest member. The *segment* S_a of A determined by $a \in A$ is defined as the set $S_a = \{x \in A | x < a\}$. Formally, an ordinal is a well-ordered set $(A, <)$ such that $S_a = a$ for all $a \in A$. The ordinals abstract the concepts expressed by "first, second, third ...".

The complimentary objects are the *cardinals*. Cardinals measure sizes, and correspond to the concepts expressed by "one, two, three, ...". The finite ordinals are the same objects as the finite cardinals. The *cardinality* of a set A is the smallest ordinal corresponding bijectively to it. In a sense, the transfinite cardinals are a subset of the transfinite ordinals. All that we need to establish here is that the ordinal ω, when viewed as a cardinal is denoted $\aleph_0$, that it is countable, and that all the other transfinite cardinals, such as $\aleph_1$, or $2^{\aleph_0}$ (which is the cardinality of the real numbers) are uncountable.

1.2 Elementary Results

The goal of this section is to prove Rado's Selection Theorem, which is a fundamental tool in the theory of infinite graphs. The first step along the way is a lemma of Zorn:

Lemma 1.1 *(Zorn's Lemma). If $(M, <)$ is a partially ordered set in which every totally ordered subset has an upper bound, then M has at least one maximal element.*

We shall not prove this set-theoretic result. Intuitively, a partially ordered set can be though of as a directed acyclic graph with each directed edge representing a pair of elements of the set that are ordered by the relation. Zorn's Lemma then states that if each path (finite or not) in the graph ends at some vertex, then there is at least one vertex in that graph that does not have a successor.

The concept of countability can be easily extended to graphs. A *countable graph* is one in which both the vertex set and the edge set are finite or

Figure 1.1: Some simple infinite graphs.

countably infinite. Consider now the simplest possible infinite graphs. Three configurations immediately present themselves: the infinite star, $K_{1,\aleph_0}$, the one-way infinite path, and the two way infinite path (see Figure 1.2). The second two are *locally finite* since every vertex has a finite degree.

Another important result attributed to D. König can be proven with Zorn's Lemma:

Lemma 1.2 *(König's Lemma). From each vertex v_0 of an infinite locally finite connected graph G, there is a one-way infinite path $(v_0, v_1, \ldots)$ such that the distance in G from each vertex v_k to v_0 is k.*

Proof: (Sketch). First prove that G has a spanning tree T rooted at v_0 such that the distances from v_0 in T are the same as those in G. Then prove that T has an infinite path. $\Box$

König's Lemma is used to extend results from finite graphs to countably infinite ones. However, it is not sufficient for uncountable graphs. In order to deal with these, we need Rado's Selection Theorem. Let E be an infinite set, and k be a positive integer. A *partial k-colouring* of E is a pair (A, c) where A is a subset of E and c maps the elements of A to $\{1, 2, \ldots, k\}$. A partial ordering can be defined on colourings by setting $(A, c) \leq (B, d)$ if $A \subseteq B$ and c is the restriction of d to A. If $\mathcal{F}$ is a collection of partial k-colourings then a partial k-colouring (B, d) is *$\mathcal{F}$-admissible* if there is no $(A, c) \in \mathcal{F}$ such that $(A, c) \leq (B, d)$.

Theorem 1.1 *(Rado's Selection Theorem). Let E be a set and let $\mathcal{F}$ be a collection of k-colourings of finite subsets of E. Then E has an $\mathcal{F}$-admissible k-colouring if and only if every finite subset of E has one.*

Proof: Sufficiency is trivial, so we will concentrate on necessity. We will construct a set M and apply Zorn's Lemma. Define M as the set of collections $\mathcal{F}'$ which satisfy the following conditions:

(i) $\mathcal{F}'$ is a set of k-colourings of finite subsets of E,

(ii) $\mathcal{F} \subseteq \mathcal{F}'$, and

(iii) all finite subsets of E have an $\mathcal{F}'$-admissible k-colouring.

M is partially ordered by the inclusion relation. In order to show that the conditions of Zorn's Lemma apply, consider some totally ordered subset M'. It will be proven that $\mathcal{F}'' = \bigcup_{\mathcal{F}' \in M'} \mathcal{F}'$, which is an upper bound of M', is in M.

$\mathcal{F}''$ clearly satisfies (i) and (ii) by construction. Let E' be some finite subset of E. If E' has no $\mathcal{F}''$-admissible k-colouring then for every (E', c), there must be a member (A', c') of some $\mathcal{F}' \in M'$ which satisfies $(E', c) \leq (A', c')$. Since there are only finitely many colourings of E', the largest such $\mathcal{F}'$ contradicts the assumption that E' has an $\mathcal{F}''$-admissible colouring. Thus Zorn's Lemma applies to M.

Let $\mathcal{F}_0$ therefore be a maximal element of M. For $i \in \{1, 2, \ldots, k\}$, let $c_i(x) = i$, and let $E_i = \{e \in E | (\{e\}, c_i) \text{is} \mathcal{F}_0 - \text{admissible}\}$. Note that each e belongs to some E_i since $\mathcal{F}_0$ satisfies (iii). Also, since $\mathcal{F}_0$ is maximal, $\mathcal{F}_0 \cup \{(\{e\}, c_i)\}$ does not satisfy (iii) (since $(\{e\}, c_i)$ is $\mathcal{F}_0$-admissible and therefore cannot be in this set already). By negating (iii), we get that E has a finite subset E' (which must include e) such that in any $\mathcal{F}_0$-admissible k-colouring of E', e receives the colour i. Furthermore, the same is true for any finite superset of E'. Now suppose the some element e belongs to two sets E_i and E_j. Then there are two finite sets E_i' and E_j' containing e in which e must always be coloured i and i respectively. But then a contradiction arises for the colouring of the set $E_i' \cup E_j'$.

Thus we have that $E_i \cap E_j = \emptyset$, whenever $i \neq j$, and so $\{E_i\}$ is a partition of E. Thus an $\mathcal{F}_0$-admissible colouring of E is obtained by assigning the colour i to all E_i for $i = 1, 2, \ldots, k$. This colouring is also $\mathcal{F}$-admissible. □

Chapter 2

Simplicial Decompositions

The idea of the simplicial decomposition is due to K. Wagner, who used it to characterize the finite maximal graphs not contractible onto K_5 or $K_{3,3}$. Simplicial decomposition has turned out to be a useful tool for infinite graphs. The primary contributors to this area have been R. Halin, G. Dirac, and R. Diestel. In this chapter, we present some of the results of Halin.

2.1 Defining Simplicial Decompositions

Let G be a graph, and $\sigma > 0$ be an ordinal. Let B_λ be an induced subgraph of G for $0 \leq \lambda < \sigma$. A *simplicial decomposition* of G is a family $F = (B_\lambda)_{\lambda<\sigma}$ satisfying the following conditions:

(S1) $G = \bigcup_{\lambda<\sigma} B_\lambda$,

(S2) $S_\mu = (\bigcup_{\lambda<\mu} B_\lambda) \cap B_\mu$ is a complete graph for $0 < \mu < \sigma$, and

(S3) no S_μ contains any B_λ for $0 \leq \lambda < \mu$.

The subgraphs S_μ are denoted this way because a complete graph is also called a *simplex*. The simplex on n vertices is denoted $S(n)$. Note that the definition covers disconnected graphs because the simplices S_μ can be $S(0)$. The members of F are called *simplicial summands*, and we define the *partial sums* $G|_\mu = \bigcup_{\lambda<\mu} B_\lambda$. The subgraphs S_μ on which the summands overlap are called the *simplices of attachment*.

A simplicial decomposition can therefore be thought of as constructing the graph G by pasting together subgraphs on overlapping simplices. See Figure 2.1 for an example. A simplicial decomposition is not necessarily unique. Simplicial decomposition is perhaps most familiar in relation to the tree decompositions of Robertson and Seymour. This connection will be discussed a little further in the next chapter.

In order to state Halin's decomposition theorem, recall that Menger's number $\mu_G(x, y)$ is the number of internally vertex-disjoint paths from x to y in G.

Figure 2.1: A simplicial decomposition of a graph.

Theorem 2.1 *Let G be a graph and n a cardinal such that $|V(G)| \geq n > \aleph_0$. Assume that G does not contain an $S(n)$ and that $\mu_G(x, y) < n$ for all pairs of non-adjacent vertices in G. Then G has a simplicial decomposition F with $\sigma = |F|$ such that σ is the initial ordinal of $|V(G)|$, and all $B \in F$ have a vertex cardinality less than n. In addition, F can be chosen such that $B_\lambda - S_\lambda$ is connected for every $\lambda < \sigma$.*

The proof of this theorem is well beyond the scope of this discussion. It is based on other results of Halin involving the concept of separation invariance.

2.2 Forbidden Configurations

In this section, we develop a characterization of some classes of graphs similar to Kurotowski's theorem. Unfortunately, some heavy definitions are required.

We define a graph G to be *prime* if it has no separating simplex. A *prime decomposition* is a simplicial decomposition F if every $B \in F$ is prime (in other words, none of the summands in F can be decomposed further). Also, a prime decomposition is called *reduced* if $B_\mu \not\subseteq B_\lambda$ whenever $\mu \neq \lambda$. We know that a prime decomposition is reduced iff all of its members are maximal. We also know:

Theorem 2.2 *If G does not contain an infinite simplex, then it admits a unique reduced prime decomposition.*

To continue with our definitions, let $\mathcal{C}$ be a class of graphs. We define $\hat{\mathcal{C}}$ as the maximal subset of $\mathcal{C}$, in the sense that for any member $G \in \hat{\mathcal{C}}$, $G + (x, y)$ is not in $\hat{\mathcal{C}}$ for any edge (x, y) not already in G.

Let G and H be graphs. If there is a bijection from $V(H)$ to a family of disjoint connected subgraphs $\{Z_v\}$ of G such that an edge (v_1, v_2) in H implies

Figure 2.2: Two graphs.

the existence of an edge between Z_{v_1} and Z_{v_2} in G, then we write $G \succ H$ and say that G is *homomorphic* to H. In other words, $G \succ H$ indicates that G contains H as a subgraph of contraction. Similarly, if G contains a subdivision of H, we write $G \succ_u H$.

If Γ is a class of finite graphs, then the *homomorphism class* $\mathcal{H}\Gamma$ is the class of all graphs G such that $G \not\succ H$ for all $H \in \Gamma$. Similarly, the *subdivision class* $\mathcal{H}_u\Gamma$ is the class of all graphs G such that $G \not\succ_u H$ for all $H \in \Gamma$.

Finally, assume Γ is a class of finite graphs. We call the set of all graphs that occur in the reduced prime decomposition of any G in $\hat{\mathcal{H}}\Gamma$ the *base* of $\hat{\mathcal{H}}\Gamma$, and denote it with $\mathcal{B}\hat{\mathcal{H}}\Gamma$.

Then we have the following two results:

Theorem 2.3 *The base $\mathcal{B}\hat{\mathcal{H}}S(5)$ consists of the (non-planar) graph W of Figure 2.2 and all prime maximal planar graphs.*

and:

Theorem 2.4 *The base $\mathcal{B}\hat{\mathcal{H}}\Gamma_4$, where Γ_4 is the class of finite 4-connected graphs, consists of $S(i)$ for $i \leq 4$ and the graphs W and P of Figure 2.2.*

Halin also presents many other results on simplicial decompositions and applications thereof, including results on connectivity, colouring problems, and normal rooted trees.

Chapter 3

Simplicial Tree Decompositions

A simplicial decomposition F is called a *simplicial tree decomposition* if it also satisfies:

(S4) Each S_μ is contained in B_λ for some $\lambda < \mu$.

If a decomposition satisfies only (S1) and (S4), it is a tree decomposition in the sense of Robertson and Seymour. For the remainder of this chapter, *prime* shall be used to describe simplicial tree decompositions rather than simplicial decompositions.

In a series of three exhaustive and exhausting papers, R. Diestel described some of the properties of simplicial and simplicial tree decompositions, and proved some results on their existence and uniqueness. In this chapter, we summarize some of his work.

3.1 Definitions and Some Simple Results

Let $F = (B_\lambda)_{\lambda<\sigma}$ be a decomposition, and let H be some subgraph of G. We denote by $\lambda(H)$ the minimum λ such that $B_\lambda \cap H \neq \emptyset$. We abbreviate $\lambda(\{x\})$ to $\lambda(x)$ (for $x \in V(G)$), and define $\Lambda(H) = \{\lambda(x) | x \in V(H)\}$. If we think of G as starting with its first summand and growing until all the B_μ have been added in, $\lambda(x)$ indicates the first stage at which x appears.

A subgraph H_1 of G is called *attached* to a subgraph H_2 of $G - H_1$ if each vertex of H_1 is adjacent to some vertex of H_2. If $S \subset G$ is a simplex which is attached to a component C of $G - S$, then the pair (C, S) is referred to as a *side (of S) in G.*

If $H \subset G$ then H is called *convex* in G if H contains all vertices of every path in G that has its end-vertices in H. The *convex hull* of H is the intersection of every convex subgraph of G containing H.

Figure 3.1: A graph, H_0, without a prime tree decomposition.

And now an important theorem which demonstrates the naturalness of simplicial decomposition, and is used to prove many other results:

Theorem 3.1 *Every B_μ and every $G|_\mu$ of a simplicial decomposition of G is convex in G.*

The proof follows from the above definition, and the simplices of attachment in the decomposition. Diestel also proves a "structure theorem" for simplicial tree decompositions, which will not be reproduced here. This theorem makes many assertions about when a maximal simplex S in G is a summand in a prime tree decomposition.

3.2 A Graph Without a Prime Tree Decomposition

In this section, we construct a graph and show that it does not admit a prime tree decomposition. Due to an earlier theorem, any such graph must have an infinite simplex. Each vertex in the graph belongs to exactly one of three infinite sets. $S = \{s_1, s_2, \ldots\}$ is an infinite simplex, and $P = \{x_1, x_2, \ldots\}$ and $Q = \{y_1, y_2, \ldots\}$ are one-way paths. H_0 is the disjoint union of S, P, and Q, and the addition of the edges (x_i, s_j) and (y_i, s_j) for $i \leq j$ (see Figure 3.2).

Let H_1 be the graph obtained from H_0 by deleting the edges of P. The finite maximal simplices in H_1 (which must be factors, due to the structure theorem) are $B'_i = H_1[x_i, s_1, \ldots, s_i]$, and $B''_i = [y_i, y_{i+1}, s_1, \ldots, s_i]$, for $i > 0$. Consider attempting to arrange these into a prime decomposition by setting $B_0 = B'_1$, $B_1 = B'_2$ and so on. Then we cannot add another summand for B_ω. Consider adding some B''_i. This violates the requirement for convexity of $H'|_{\omega+1}$ because there is a path from y_i to y_{i+1} which uses vertices y_{i-1} and y_{i+2}.

3.3 Existence of Prime Decompositions

In the second of three papers, R. Diestel discusses the existence of a simplicial tree decomposition into primes. We will state the theorem proved in this paper, but omit the proof.

Recall the definition of a side (C, S) in G. If $S' \subset S$ and X is an induced subgraph subgraph of G such that $S' \subset X$ and $X \cap C \neq \emptyset$, then X is an *extension* of S' into C.

Theorem 3.2 *A countable graph G admits a simplicial tree decomposition into primes iff G satisfies:*

- *If (C, S) is a side in G and C' is a component of $G - S$ and $C' \neq C$ then $S[C']$ has a prime extension into C.*

A simple and elegent proof of this theorem is not known at this time.

3.4 Uniqueness of Prime Decompositions

Finally, the third paper of Diestel concerns the uniqueness of a simplicial tree decomposition into primes. We will not state this theorem here, but rather attempt to convey some intuition to the reader. It is stated in the paper that the ordering of the summands in a decomposition is clearly not necessarily unique. This is intuitively obvious due to the tree structure of a decomposition.

It is also not the case that the summands are unique. It is possible for a graph to have "superfluous" prime summands. However, every factor is present in some decomposition.

The one aspect of a decomposition that is truly unique is the set of the simplices of attachment.

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