The Rendering Equation

Kajiya '86

Michael Tao

University of Toronto

Since Derek made slides...

The Rendering Equation

$$I(x, x') = g(x, x') \left[\epsilon(x, x') + \int_S \rho(x, x', x'') I(x', x'') dx'' \right]$$

- I: intensity of light from x to x'
- g: geometry
- ϵ: emission
- ρ: intensity of light from x'' to x from x'

Limitations

- Time-averaged transport intensity
  - No phasees
    - No diffraction
- Homogeneous refractive index of base medium
  - Could let g handle this (single scattering)
  - Could solve this equation volumetrically (Eikonal)
  - Could solve this equation with sparse paths (path integration)

Stoichiometry?

- How should we set I (that didn't exist before this paper)?
- Isn't it more natural to talk about angles around a point?
- Use stoichiometry to connect with standard radiometic intensity!

Stoichiometry

$$dE = I(x, x')\,dt\,dx\,dx'$$

- $dE \sim joule$
- $dx, dx' \sim m^2$
- $dt \sim sec$
- $\Rightarrow I(x, x') \sim \frac{joule}{m^4 sec}$

Stoichiometry

$$g(x, x') \sim \frac{Visible(x, x')}{r^2}$$

- Unitless!

Stoichiometry

$$dE = \frac{1}{r^2}\epsilon(x, x')dtdxdx'$$

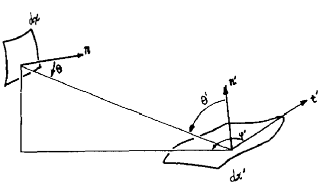
- $dE \sim joule$
- $dx, dx' \sim m^2$
- $dt \sim sec$
- $\Rightarrow I(x, x') \sim \frac{joule}{m^2 sec}$

$$dE = \frac{1}{r^2}\rho(x, x', x'')I(x', x'')dtdxdx'dx''$$

- $dE \sim joule$
- $dx, dx', dx'' \sim m^2$
- $dt \sim sec$
- $I(x, x') \sim \frac{joule}{m^4 sec}$
- $\Rightarrow$ unitless!

Radiometric Intensity

- $i(\theta, \phi)$
  - per time (dt)
  - per unit projected area (dx_p')
  - per unit of solid angle ($d\omega$)



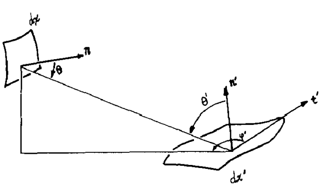
$$dE = i(\theta', \phi') d\omega dx_p' dt$$

Radiometric Intensity

- After defining maps between points and angles...

$$
\begin{aligned}
r &= \|x - x'\| \\
dx'_p &= dx' \cos\theta \\
\cos\theta &= \frac{1}{r}\langle n, x - x'\rangle \\
\cos\theta' &= \frac{1}{r}\langle n', x - x'\rangle \\
\cos\phi' &= \frac{1}{r}\langle t', x - x'\rangle
\end{aligned}
$$

$$dE = i(\theta', \phi')\frac{\cos\theta\cos\theta'}{r^2}dtdxdx'$$

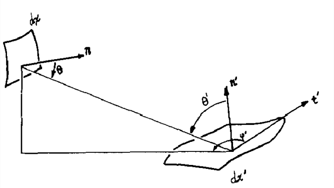


Radiometric Intensity to Intensities

- After defining maps between points and angles...

$$I(x, x')dtdxdx' = dE = i(\theta', \phi')\frac{\cos\theta\cos\theta'}{r^2}dtdxdx'$$

$$I(x, x') = i(\theta', \phi')\frac{\cos\theta\cos\theta'}{r^2}$$

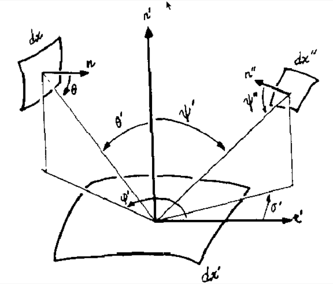


And everything else

- After some more boring algebraic manipulation with several variables...

$$\epsilon(x, x') = \epsilon(\theta', \phi') \cos\theta \cos\theta'$$

$$\rho(x, x', x'') = \rho(\theta', \phi', \psi', \sigma') \cos\theta \cos\theta'$$



So what?

- The basic rendering equation is really elegant and now we have a bunch of cos everywhere, which is important
- Now we can try solving it maybe?

Neumann Series

- As the equation is an affine transformation we can rewrite it as

$$I = g\epsilon + gMI$$

- We can solve a steadystate by...

$$(1 - gM)I = g\epsilon$$

- Taylor expansion of hte above provides

$$(1 - gM)^{-1} g\epsilon = g \left(\sum_{i=0}^{\infty} [Mg]^i \right) \epsilon$$

- Intuitively,
  - $i = 0$ is light directly emitted
  - $i = 1$ is light bounced once
  - $i = n$ is what?

Utah Approximation

- If we have a precomputed illumination ϵ_0

$$I = g\epsilon + gM\epsilon_0$$

- this is too easy?
- Well Whitted proposed something slightly more general

$$I = g\epsilon + g\left(\sum_{i=1}^{\infty} [M_0 g]^i\right)\epsilon_0$$

  - M_0 is delta functions with a cosine...

- What if there is no angular dependence?

$$\rho(\theta', \phi', \psi', \sigma') = \rho_0$$

- Well radiosity $dB_r(x)$ (a reduced I turns into

$$dB_r(x') = dx' \rho_0 \pi H(x')$$

  - for H hemispherical incident energy per time per area
- With emission we get

$$dB_r(x') = \pi[\epsilon_0 + \rho_0 H(x')]dx'$$

- By energy conservation, $\|M\| \leq 1$.
- Consider

$$x + a + \sum_{k=1}^{\infty} M^k a$$

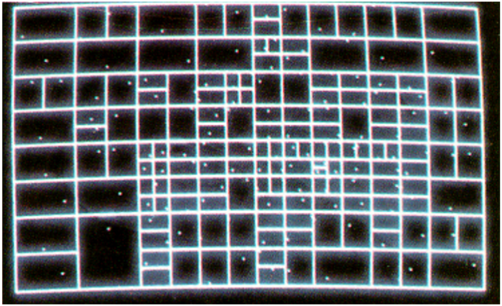
- We can construct a probability of a path of length n, $\omega \in S^n$ to obtain the contribution from σ_1 to x_ℓ by

$$x_\ell = \left(\prod_{i=0}^{\ell} m_{\sigma_{i-1},\sigma_i} \right) a_{\sigma_i} \frac{1}{p(\omega)}$$

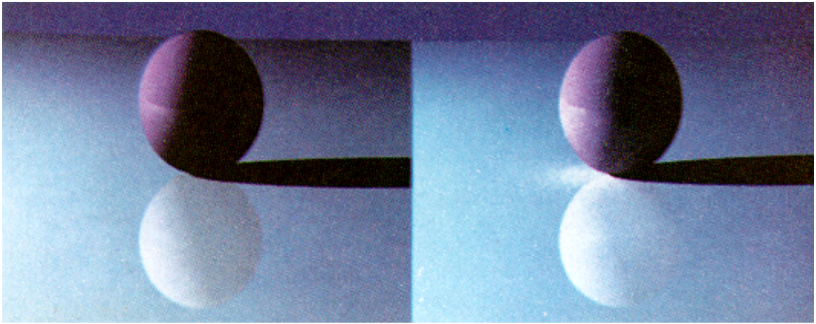
- Apply splitting to say

$$p(\omega) = p(\omega_0) \prod_{0}^{\ell} p(\omega_i, \omega_{i+1})$$

- To not oversample they construct a hierarchy with one sample per cell



Results?



Results?