# Nested Radicals

## And Other Infinitely Recursive Expressions

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# Outline

# 1. Introduction

## Examples of Infinitely Recursive Expressions

### Series

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots$$

### Infinite Products

$$\pi/2 = \frac{2}{1} \times \frac{2}{3} \times \frac{4}{3} \times \frac{4}{5} \times \frac{6}{5} \times \cdots$$

### Continued Fractions

$$e - 1 = 1 + \cfrac{2}{2 + \cfrac{3}{3+\cfrac{4}{4+\cfrac{5}{5+\ldots}}}}$$

$$4/\pi = 1 + \cfrac{1}{2 + \cfrac{3^2}{2+\cfrac{5^2}{2+\cfrac{7^2}{2+\ldots}}}}$$

Infinitely Nested Radicals (or Continued Roots)

$$K = \sqrt{1 + \sqrt{2 + \sqrt{3 + \dots}}}$$

Exponential Ladders (or Towers)

$$2 = (\sqrt{2})^{(\sqrt{2})^{(\sqrt{2})^{\cdots}}}$$

Hybrid Forms

$$4 = 2^{\sqrt{2^{\sqrt{2^{\sqrt{2^{\cdots}}}}}}}$$

$$\frac{1}{2} = \frac{1}{\frac{1}{\frac{1}{\cdots}+1+\frac{1}{\cdots}} + 1 + \frac{1}{\frac{1}{\cdots}+1+\frac{1}{\cdots}}}$$

Questions:

- Does the expression converge ? Are there *tests*, or necessary/sufficient conditions for convergence ? Examples:
  - For series,
    - Terms must go to zero
    - d'Alembert-Cauchy Ratio Test, Cauchy *n*th Root Test, Integral Test, ...
  - For infinite products,
    - Terms must go to a value in (-1,1]
  - For infinitely nested radicals,
    - Terms can grow ! (But how fast ?)

- What does the expression converge to ? Are there formulae or identities we can use to evaluate the limit? Example: when $-1 < r < 1$,

$$\frac{a}{1-r} = a + ar + ar^2 + ar^3 + \ldots$$

## 2.1 Constant Term Expansions

Assume that

$$\sqrt{a + b\sqrt{a + b\sqrt{a + \ldots}}}$$

converges when $a \geq 0$ and $b \geq 0$, and let $L$ be the limit. Then

$$L = \sqrt{a + b\sqrt{a + b\sqrt{a + \ldots}}}$$

$$L = \sqrt{a + bL}$$

$$L^2 - bL - a = 0$$

$$L = \frac{b + \sqrt{b^2 + 4a}}{2}$$

Hence

$$\sqrt{a + b\sqrt{a + b\sqrt{a + \ldots}}} = \frac{b + \sqrt{b^2 + 4a}}{2}$$

Observation: when $a = 0$, we get

$$\sqrt{0 + b\sqrt{0 + b\sqrt{0 + \dots}}} = \frac{b + \sqrt{b^2 + 4(0)}}{2}$$

$$\sqrt{b\sqrt{b\sqrt{b\sqrt{\dots}}}} = b$$

This makes sense since

$$\begin{aligned}
\sqrt{b\sqrt{b\sqrt{b\sqrt{\dots}}}} &= \sqrt{b}\sqrt{\sqrt{b}}\sqrt{\sqrt{\sqrt{b}}}\dots \\
&= b^{\frac{1}{2}}b^{\frac{1}{4}}b^{\frac{1}{8}}\dots \\
&= b^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\dots} \\
&= b^1
\end{aligned}$$

Similarly, assume that

$$a + \frac{b}{a + \frac{b}{a + \frac{b}{a + \ldots}}}$$

converges when $a > 0$ and $b \geq 0$, and let $L$ be the limit. Then

$$L = a + \frac{b}{a + \frac{b}{a + \frac{b}{a + \ldots}}}$$

$$L = a + \frac{b}{L}$$

$$L^2 - aL - b = 0$$

$$L = \frac{a + \sqrt{a^2 + 4b}}{2}$$

Hence

$$a + \cfrac{b}{a + \cfrac{b}{a + \cfrac{b}{a + \dots}}} = \frac{a + \sqrt{a^2 + 4b}}{2}$$

But as we saw earlier,

$$\sqrt{a + b\sqrt{a + b\sqrt{a + \dots}}} = \frac{b + \sqrt{b^2 + 4a}}{2}$$

Therefore,

$$\sqrt{a + b\sqrt{a + b\sqrt{a + \dots}}} = b + \cfrac{a}{b + \cfrac{a}{b + \cfrac{a}{b + \dots}}} = \frac{b + \sqrt{b^2 + 4a}}{2}$$

In addition, setting $a = b = 1$, we get

$$\sqrt{1 + \sqrt{1 + \sqrt{1 + \dots}}} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \dots}}} = \frac{1 + \sqrt{5}}{2}$$

which is equal to the golden ratio $\phi$.

Now assume that

$$\sqrt[n]{a+b\sqrt[n]{a+b\sqrt[n]{a+\dots}}}$$

converges when $a \geq 0$ and $b \geq 0$, and let $L$ be the limit. Then

$$L = \sqrt[n]{a+b\sqrt[n]{a+b\sqrt[n]{a+\dots}}}$$

$$L = \sqrt[n]{a+bL}$$

$$L^n - bL - a = 0$$

Let $\alpha = a/L^n$ and $\beta = b/L^{n-1}$. Then $a = \alpha L^n$, $b = \beta L^{n-1}$ and

$$L^n - \beta L^n - \alpha L^n = 0$$

$$1 - \beta - \alpha = 0$$

$$\beta = 1 - \alpha$$

yielding

$$L = \sqrt[n]{\alpha L^n + \beta L^{n-1}\sqrt[n]{\alpha L^n + \beta L^{n-1}\sqrt[n]{\alpha L^n + \dots}}}$$

## 2.2 Identity Transformations

Pushing terms through radicals,

$$
\begin{aligned}
\frac{b+\sqrt{b^2+4a}}{2} &= \sqrt{a+b\sqrt{a+b\sqrt{a+b\sqrt{a+\ldots}}}} \\
&= \sqrt{a+\sqrt{ab^2+b^3\sqrt{a+b\sqrt{a+\ldots}}}} \\
&= \sqrt{a+\sqrt{ab^2+\sqrt{ab^6+b^7\sqrt{a+\ldots}}}} \\
&= \sqrt{a+\sqrt{ab^2+\sqrt{ab^6+\sqrt{ab^{14}+\ldots}}}} \\
&= \sqrt{\left(\frac{a}{b^2}\right)b^2+\sqrt{\left(\frac{a}{b^2}\right)b^4+\sqrt{\left(\frac{a}{b^2}\right)b^8+\sqrt{\left(\frac{a}{b^2}\right)b^{16}+\ldots}}}}
\end{aligned}
$$

Set $\alpha = a/b^2$. Then

$$
\begin{aligned}
\sqrt{\alpha b^2+\sqrt{\alpha b^4+\sqrt{\alpha b^8+\sqrt{\alpha b^{16}+\ldots}}}} &= \frac{b+\sqrt{b^2+4a}}{2} \\
&= \frac{b+\sqrt{b^2+4\alpha b^2}}{2} \\
&= \frac{b}{2}\left(1+\sqrt{1+4\alpha}\right)
\end{aligned}
$$

Setting $\alpha = 2$, $b = 1/2$,

$$\sqrt{\frac{2}{2^2} + \sqrt{\frac{2}{2^4} + \sqrt{\frac{2}{2^8} + \sqrt{\frac{2}{2^{16}} + \ldots}}}} = 1$$

$$\sqrt{\frac{2}{2^1} + \sqrt{\frac{2}{2^2} + \sqrt{\frac{2}{2^4} + \sqrt{\frac{2}{2^8} + \ldots}}}} = \sqrt{2}$$

This can be rewritten as

$$2^{1-2^{-1}} = \sqrt{2^{1-2^0} + \sqrt{2^{1-2^1} + \sqrt{2^{1-2^2} + \ldots}}}$$

And generalized to

$$2^{1-2^k} = \sqrt{2^{1-2^{k+1}} + \sqrt{2^{1-2^{k+2}} + \sqrt{2^{1-2^{k+3}} + \ldots}}}$$

Letting $k \to -\infty$,

$$2 = \sqrt{\ldots + \sqrt{2^{1-2^{-1}} + \sqrt{2^{1-2^0} + \sqrt{2^{1-2^1} + \ldots}}}}$$

Transformations for "pushing" terms through radicals:

$$\sqrt{a_0 + b_0\sqrt{a_1 + b_1\sqrt{a_2 + b_2\sqrt{a_3 + \dots}}}}$$

$$= \sqrt{a_0 + \sqrt{a_1 b_0^2 + \sqrt{a_2 b_1^2 b_0^4 + \sqrt{a_3 b_2^2 b_1^4 b_0^8 + \dots}}}}$$

$$\sqrt[n]{a_0 + b_0\sqrt[n]{a_1 + b_1\sqrt[n]{a_2 + b_2\sqrt[n]{a_3 + \dots}}}}$$

$$= \sqrt[n]{a_0 + \sqrt[n]{a_1 b_0^n + \sqrt[n]{a_2 b_1^n b_0^{n^2} + \sqrt[n]{a_3 b_2^n b_1^{n^2} b_0^{n^3} + \dots}}}}$$

## 2.3 Generation of Identities Using Recurrences

Srinivasa Ramanujan (1887-1920)

$$1 - 5\left(\frac{1}{2}\right)^3 + 9\left(\frac{1\times 3}{2\times 4}\right)^3 - 13\left(\frac{1\times 3\times 5}{2\times 4\times 6}\right)^3 + \ldots = 2/\pi$$

$$\frac{1}{1+\frac{e^{-2\pi}}{1+\frac{e^{-4\pi}}{1+\frac{e^{-6\pi}}{1+\ldots}}}} = \left(\sqrt{\frac{5+\sqrt{5}}{2}} - \frac{\sqrt{5}+1}{2}\right) e^{(2\pi/3)}$$

$$\left(1 + \frac{1}{1\times 3} + \frac{1}{1\times 3\times 5} + \ldots\right) + \frac{1}{1+\frac{1}{1+\frac{2}{1+\frac{3}{1+\ldots}}}} = \sqrt{\frac{\pi e}{2}}$$

Problem:

$$? = \sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + 4\sqrt{1 + \ldots}}}}$$

Ramanujan claimed:

$$x + n = \sqrt{n^2 + x\sqrt{n^2 + (x+n)\sqrt{n^2 + (x+2n)\sqrt{\ldots}}}}$$

Setting $n$=1 and $x$=2 we find

$$3 = \sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + 4\sqrt{1 + \ldots}}}}$$

Notice

$$[a+b] = \sqrt{b^2 + a^2 + 2ab} = \sqrt{b^2 + a[a+b+b]}$$

Expanding the square-bracketed portions,

$$\begin{aligned}
[x+n] &= \sqrt{n^2 + x[x+n+n]} \\
&= \sqrt{n^2 + x\sqrt{n^2 + (x+n)[x+2n+n]}} \\
&= \sqrt{n^2 + x\sqrt{n^2 + (x+n)\sqrt{n^2 + (x+2n)[x+3n+n]}}} \\
&\vdots \\
&= \sqrt{n^2 + x\sqrt{n^2 + (x+n)\sqrt{n^2 + (x+2n)\sqrt{\ldots}}}}
\end{aligned}$$

Basic Idea:

- Find a "telescoping" recurrence relation
- Use it to generate an infinitely recursive expression
- Hope that it converges (!)

Consider a more familiar recurrence relation

$$\left[\frac{1}{k}\right] = \frac{1}{k(k+1)} + \left[\frac{1}{k+1}\right]$$

Expanding the square-bracketed portions,

$$\begin{aligned}
\left[\frac{1}{n}\right] &= \frac{1}{n(n+1)} + \left[\frac{1}{n+1}\right] \\
&= \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} + \left[\frac{1}{n+2}\right] \\
&\ \ \vdots \\
&= \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+2)(n+3)} + \ldots
\end{aligned}$$

In this case, the infinite expansion is valid.

Consider the recurrence

$$\left[2^{1-2^k}\right] = \sqrt{2^{1-2^{k+1}} + \left[2^{1-2^{k+1}}\right]}$$

which expands into

$$\left[2^{1-2^k}\right] = \sqrt{2^{1-2^{k+1}} + \sqrt{2^{1-2^{k+2}} + \sqrt{2^{1-2^{k+3}} + \ldots}}}$$

Next, consider the recurrence

$$\left[1 + 2^{-2^{k+1}}\right] = \sqrt{2^{1-2^{k+1}} + \left[1 + 2^{-2^{k+2}}\right]}$$

which expands into

$$\left[1 + 2^{-2^{k+1}}\right] = \sqrt{2^{1-2^{k+1}} + \sqrt{2^{1-2^{k+2}} + \sqrt{2^{1-2^{k+3}} + \ldots}}}$$

How can two identities have the same right hand side but different left hand sides ? Answer: in the second identity, the infinite expansion is not valid.

Another example (this time of a valid expansion). The recurrence

$$[n! + (n+1)!] = \sqrt{n!^2 + n!\,[(n+1)! + (n+2)!]}$$

expands into

$$[n! + (n+1)!] = \sqrt{n!^2 + n!\sqrt{(n+1)!^2 + (n+1)!\sqrt{(n+2)!^2 + \ldots}}}$$

Recalling that $\Gamma(k+1) = k!$ for natural $k$, we can generalize to

$$[\Gamma(x) + \Gamma(x+1)] = \sqrt{\Gamma^2(x) + \Gamma(x)\sqrt{\Gamma^2(x+1) + \Gamma(x+1)\sqrt{\ldots}}}$$

## 3. General Forms

Consider a "continued power" of the form

$$a_0 + b_0(a_1 + b_1(a_2 + b_2(a_3 + \ldots)^{p_2})^{p_1})^{p_0}$$

Setting $p_j = 1$ and $b_j = 1$, we get a series

$$a_0 + a_1 + a_2 + a_3 + \ldots$$

Setting $p_j = 1$ and $a_j = 0$, we get an infinite product

$$b_0 b_1 b_2 b_3 \ldots$$

Setting $p_j = -1$, we get a continued fraction

$$a_0 + \frac{b_0}{a_1 + \frac{b_1}{a_2 + \frac{b_2}{a_3 + \ldots}}}$$

Setting $p_j = 1$ and $b_j = 1/c_j$, we get an ascending continued fraction

$$a_0 + \frac{a_1 + \frac{a_2 + \frac{a_3 + \ldots}{c_2}}{c_1}}{c_0}$$

Setting $p_j = 1/n$, we get a nested radical

$$a_0 + b_0 \sqrt[n]{a_1 + b_1 \sqrt[n]{a_2 + b_2 \sqrt[n]{a_3 + \ldots}}}$$

Setting $p_j = -1/n$, we get a hybrid form

$$a_0 + \cfrac{b_0}{\sqrt[n]{a_1 + \cfrac{b_1}{\sqrt[n]{a_2 + \cfrac{b_2}{\sqrt[n]{a_3 + \dots}}}}}}$$

Observation: series, infinite products, continued fractions and nested radicals are all special cases of this generalized "continued power" form !

Question: can another general form be found for which exponential ladders are also a special case ?

We can imagine constructing the expression

$$a_0 + b_0(a_1 + b_1(a_2 + b_2(a_3 + \ldots)^{p_2})^{p_1})^{p_0}$$

by starting with a "seed" term and repeating the following steps:

- Raise to the exponent $p_j$
- Multiply by $b_j$
- Add $a_j$

Of these 3 operations, only the first is non-commutative. What if we change the ordering of the operands in the first step ? Then we would constuct an expression like

$$a_0 + b_0 p_0^{a_1 + b_1 p_1^{a_2 + b_2 p_2^{a_3 + \cdots}}}$$

Setting $a_j = 0$ and $b_j = 1$, we get an exponential ladder

$$p_0^{p_1^{p_2^{p_3^{\cdots}}}}$$

What other things can we generalize ?

- Identities. Example (constant term expansion):

$$L = \sqrt[n]{\alpha L^n + \beta L^{n-1}\sqrt[n]{\alpha L^n + \beta L^{n-1}\sqrt[n]{\alpha L^n + \ldots}}}$$

  becomes

  $L = (\alpha L^{1/p}+\beta L^{1/p-1}(\alpha L^{1/p}+\beta L^{1/p-1}(\alpha L^{1/p}+\ldots)^p)^p)^p$

  where $\beta = 1 - \alpha$.

- Recurrences. Example:

$$\left[2^{1-2^k}\right] = \sqrt{2^{1-2^{k+1}} + \left[2^{1-2^{k+1}}\right]}$$

  becomes

$$\left[2^{\frac{p^k-1}{p^{k-1}-p^k}}\right] = \left(2^{\frac{p^{k+1}-1}{p^k-p^{k+1}}} + \left[2^{\frac{p^{k+1}-1}{p^k-p^{k+1}}}\right]\right)^p$$

- Transformations. Example:

$$\sqrt[n]{a_0 + b_0 \sqrt[n]{a_1 + b_1 \sqrt[n]{a_2 + b_2 \sqrt[n]{a_3 + \ldots}}}}$$

$$= \sqrt[n]{a_0 + \sqrt[n]{a_1 b_0^n + \sqrt[n]{a_2 b_1^n b_0^{n^2} + \sqrt[n]{a_3 b_2^n b_1^{n^2} b_0^{n^3} + \ldots}}}}$$

becomes

$$(a_0 + b_0(a_1 + b_1(a_2 + b_2(a_3 + \ldots)^p)^p)^p)^p$$

$$= (a_0 + (a_1 b_0^{p^{-1}} + (a_2 b_1^{p^{-1}} b_0^{p^{-2}} + (a_3 b_2^{p^{-1}} b_1^{p^{-2}} b_0^{p^{-3}} + \ldots)^p)^p)^p)^p$$

- Convergence Tests. Example: Is there a generalized ratio test like the one used with series ?

# 4. Selected Results from Literature

## Infinite Products

A) If $-1 < x < 1$, then

$$\prod_{j=0}^{\infty}\left(1+x^{2^j}\right)=\frac{1}{1-x}$$

Incidentally, this identity can be generated with the recurrence

$$\left[\frac{1}{1-x}\right]=(1+x)\left[\frac{1}{1-x^2}\right]$$

B) If $F_n = 2^{2^n}+1 =$ the $n$th Fermat number, then

$$\prod_{n=0}^{\infty}\left(1-\frac{1}{F_n}\right)=\frac{1}{2}$$

C) If the factors of an infinite product all exceed unity by small amounts that form a convergent series, then the infinite product also conveges.

Exponential Ladders

If $0.06599 \approx e^{-e} \le x \le e^{1/e} \approx 1.44467$, then

$$x^{x^{x^{x^{\cdots}}}}$$

converges to a limit $L$ such that $L^{1/L} = x$.

Herschfeld's Convergence Theorem (restricted), published 1935. When $x_n > 0$ and $0 < p < 1$, the expression

$$\lim_{k\to\infty} x_0 + (x_1 + (\ldots + (x_k)^p \ldots)^p)^p$$

converges if *and only if* $\{x_n^{p^n}\}$ is bounded.

Special case: $p = 1/2$. Then

$$\lim_{k\to\infty} x_0 + \sqrt{x_1 + \sqrt{\ldots + \sqrt{x_k}}}$$

converges if and only if $\{x_n^{2^{-n}}\}$ is bounded.

"Souped-up" ratio test (due to Dixon Jones, 1988). When $x_n > 0$ and $p > 1$, the continued power

$$\lim_{k\to\infty} x_0 + (x_1 + (\ldots + (x_k)^p \ldots)^p)^p$$

converges if

$$\frac{x_{n+1}^p}{x_n} \le \frac{(p-1)^{p-1}}{p^p}$$

for all sufficiently large $n$.

Observation: as $p \to 1$, we *almost* get back d'Alembert's ratio test for series.