Wavelets – An Introduction

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Abstract

Wavelets are used in a wide range of applications such as signal analysis, signal compression, finite element methods, differential equations, and integral equations. In the following we will discuss the limitations of traditional basis expansions and show why wavelets are in many cases more efficient representations. A mathematical treatment of second generation wavelets as well as an example will be provided.

1 What are wavelets . . . and why do we care?

Traditional basis expansions such as the Fourier transform and the Laplace transform have proven to be indispensable in many domains. In the last decades it has however been recognized that different limitations hamper the practicality of these representations:

(L1) *Localization in space* The Fourier transform is localized in frequency but the global support of the basis functions prevents a localization in space[1]. For many applications in particular the local behaviour of signals is of interest.

(L2) *Faster transform algorithms* In recent years the advance of data acquisition technology outpaced the available computing power significantly making the Fast Fourier Transform with its $\mathcal{O}(n \log n)$ complexity a bottleneck in many applications.

(L3) *More flexibility* Traditional basis expansions provide no or almost no flexibility. It is therefore usually not possible to adapt a representation to the problem at hand. An important reason for this lack of flexibility is the orthogonal nature of traditional basis expansions.

(L4) *Arbitrary domains* Traditional basis representations can only represent functions defined of Euclidean spaces $\mathbb{R}^n$ [2]. Many real-world problems have embeddings $X \subset \mathbb{R}^n$ as domain and it is desirable to have a representation which can be easily adapted for these spaces.

(L5) *Weighted measures and irregularly sampled data* Traditional transforms can usually not be employed on spaces with weighted measures or when the input data is irregularly sampled.

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[1] The frequency localization of the Fourier transform refers to the fact that every Fourier basis function captures characteristics of the input signal in a limited frequency band. Space localization refers to a limited effective support of the basis functions in the primary domain, for audio signals, for example, the primary or "space" domain is time.

[2] A notable exception is the sphere where, for example, Spherical Harmonics [10] provide a basis.

These limitations motivated the development of *wavelets*. Many different fields such as applied mathematics, physics, signal processing, and computer science provided contributions and today both a thorough mathematical theory and fast and practical algorithms exist.

An important distinction between traditional basis expansions and wavelets is that there is not a single set of basis functions that defines a wavelet. Instead, the members of a family of representations with vastly different properties are denoted as wavelets. Common to all of them are three properties:

(P1) The sequence $\{f_k\}_{k=1}^m$ forms a basis or a frame[3] of L_p[4].

(P2) The elements of $\{f_k\}_{k=1}^m$ are localized in both space and frequency.

(P3) Fast algorithms for the analysis, synthesis, and processing of signals in its basis representation exist.

These three properties – and the flexibility they leave – are the key to the efficiency and versatility of wavelets.

Some of the first non-trivial wavelets that have been developed are the Daubechies wavelet [4] and the Meyer wavelet [11]. These, and most other wavelets developed in the 1980s, are *first generation wavelets* whose construction requires the Fourier transform and whose basis functions have to be (dyadic) scales and translates of one particular *mother basis function*[5] (cf. Section 3). The limitations L3 to L5 thus still apply for first generation wavelets. The work by Mallat and Sweldens overcame these restrictions and led to the development of *second generation wavelets* which will be discussed in more detail in the following section.

Wavelets can be categorized into *discrete* (DWT) and *continuous* (CWT) wavelet transforms. To speak in broad terms, the basis functions of DWTs are defined over a discrete space which becomes continuous only in the limit case, whereas the basis functions of CWTs are continuous but require discretization if they are to be used on a computer; see for example the book by Antoine et al. [1] for a more detailed discussion of the differences. In signal compression applications mostly discrete wavelets are employed, whereas for signal analysis typically continuous wavelets are used.

2 Second Generation Wavelets

In this section a mathematical characterization of second generation wavelets will be provided. See the paper by Sweldens [17] or the thesis by Lessig [9] for a more comprehensive treatment.

Second generation wavelets permit the representation of functions in L_2, the space of functions with finite energy [6], in a very general setting $L_2 \equiv L_2(X, \Sigma, \mu)$, where $X \subseteq \mathbb{R}^n$ is a spatial domain, Σ denotes a σ-algebra defined over X, and μ is a (possibly weighted) measure on Σ[7]. The inner product defined over X will be denoted as $\langle \cdot, \cdot \rangle$. A *multiresolution analysis* $\mathcal{M} = \{V_j \subset L_2 \mid j \in \mathcal{J} \subset \mathbb{Z}\}$ consisting of a sequence of nested subspaces V_j on different levels j is employed to define the basis functions. $\mathcal{M}$ satisfies

[3] A frame is an overcomplete representation, that is some basis functions f_i can be represented as linear combination of other basis functions. See the book by Christensen [2] for more details.

[4] In the following we will only consider the space L_2 of functions with finite energy.

[5] See the book by Chui [3] for a more detailed discussion.

[6] In engineering and many other disciplines, "finite energy" is often used synonymously with "square-integrable", that is the ℓ_2 norm of all functions in the space has to be finite.

[7] For first generation wavelets, $X = \mathbb{R}^n$ and μ is the Haar-Lebesgue measure [5].

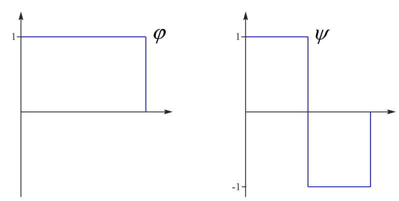

Figure 1: Father scaling basis function φ and wavelet mother basis function ψ for the Haar basis.

1. $V_j \subset V_{j+1}$.
2. $\bigcup_{j\in\mathcal{J}} V_j$ is dense in L_2.
3. For every $j \in \mathcal{J}$, a basis of V_j is given by *scaling functions* $\{\varphi_{j,k} \mid k \in \mathcal{K}(j)\}$.

The index set $\mathcal{K}(j)$ is defined over all basis functions on level j. Next to the primary multiresolution analysis $\mathcal{M}$, a *dual multiresolution analysis* $\tilde{\mathcal{M}} = \{\tilde{V}_j \subset L_2 \mid j \in \mathcal{J} \subset \mathbb{Z}\}$ formed by dual spaces $\tilde{V}_j$ exists, and a basis of the $\tilde{V}_j$ is given by dual scaling functions $\{\tilde{\varphi}_{j,k} \mid k \in \mathcal{K}(j)\}$. The primary and dual scaling functions are *biorthogonal*

$$\langle \varphi_{j,k}, \tilde{\varphi}_{j,k'} \rangle = \delta_{k,k'}.$$

The nested structure of the spaces $V_j \subset V_{j+1}$ implies the existence of difference spaces W_j with $V_j \oplus W_j = V_{j+1}$. The W_j are spanned by sets of *wavelet basis functions* $\{\psi_{j,m} \mid m \in \mathcal{M}(j)\}$. Analogous to the spaces $\tilde{V}_j$, dual wavelet spaces $\tilde{W}_j$ with $\tilde{V}_j \oplus \tilde{W}_j = \tilde{V}_{j+1}$ exist. These are spanned by dual wavelet basis functions $\{\tilde{\psi}_{j,m} \mid m \in \mathcal{M}\}$. The primary and dual wavelet basis functions *on all levels* are biorthogonal

$$\langle \psi_{j,m}, \psi_{j',m'} \rangle = \delta_{j,j'} \delta_{k,k'}.$$

For all levels j, the spaces V_j and W_j are subspaces of V_{j+1}. This implies the existence of refinement relationships

$$\varphi_{j,k} = \sum_{l\in\mathcal{L}(j,k)} h_{j,k,l}\, \varphi_{j,l} \quad \text{and} \quad \psi_{j,m} = \sum_{l\in\mathcal{L}(j,m)} g_{j,m,l}\, \varphi_{j,l}. \tag{1}$$

The $h_{j,k,l}$ and $g_{j,m,l}$ are *scaling filter coefficients* and *wavelet filter coefficients*, respectively. The index sets $\mathcal{L}(j,k)$ and $\mathcal{L}(j,m)$ are defined as

$$\mathcal{L}(j,k) = \{l \in \mathcal{K}(j+1) \mid h_{j,k,l} \neq 0\} \qquad \mathcal{L}(j,m) = \{l \in \mathcal{K}(j+1) \mid g_{j,m,l} \neq 0\}$$

and can be augmented by index sets

$$\mathcal{K}(j,l) = \{k \in \mathcal{K}(j) \mid h_{j,k,l} \neq 0\} \qquad \mathcal{M}(j,l) = \{m \in \mathcal{M}(j) \mid g_{j,m,l} \neq 0\}.$$

In the following, unless stated otherwise, l is assumed to run over $\mathcal{L}(j,k)$ or $\mathcal{L}(j,m)$, k over $\mathcal{K}(j,l)$, and m over $\mathcal{M}(j,l)$. Refinement relationships analogous to Eq. 1 hold for the $\tilde{\varphi}_{j,k}$ and $\tilde{\psi}_{j,m}$ with dual filter coefficients $\tilde{h}_{j,k,l}$ and $\tilde{g}_{j,k,m}$. The corresponding index sets, for example $\tilde{\mathcal{L}}(j,mk)$, are defined analogous to the index sets for the primary basis functions.

The discrete nature of second generation wavelets results from the definition of the basis functions over a *partition* $\mathcal{S}$. A set of measurable subsets $\{S_{j,n} \mid j \in \mathcal{J}, n \in \mathcal{N}\}$, with $\mathcal{N}(j)$ being an index set defined over all $S_{j,n}$ on level j, is a partition iff:

1. $\forall j \in \mathcal{J}$: clos$\bigcup_{n\in\mathcal{N}(j)} S_{j,n} = X$ and the union is disjoint; that is for fixed j the $S_{j,n}$ form a simple cover of X.

2. $\mathcal{N}(j) \subset \mathcal{N}(j+1)$.
3. $S_{j,n+1} \subset S_{j,n}$.
4. For fixed $n_0 \in \mathcal{N}(j)$, $\bigcap_{j>j_0} S_{j,n}$ is a set containing one point.

Subdivision schemes for embeddings $X \subset \mathbb{R}^n$ are well-known examples for partitions.

The properties of a wavelet basis can be related back to the filter coefficients. The biorthogonality of the basis functions, for example, can be written as

$$\begin{array}{cc} \sum_l g_{j,m,l}\,\tilde{g}_{j',m',l} = \delta_{j,j'}\delta_{m,m'} & \sum_l h_{j,k,l}\,\tilde{h}_{j,k',l} = \delta_{k,k'} \\ \sum_l h_{j,k,l}\,\tilde{g}_{j',m,l} = 0 & \sum_l \tilde{h}_{j,k,l}\,g_{j',m,l} = 0. \end{array} \tag{2}$$

A set of scaling and wavelet basis functions provides perfect reconstruction if Eq. 2 holds and

$$\sum_k h_{j,k,l}\,\tilde{h}_{j,k,l} + \sum_m g_{j,m,l}\,\tilde{g}_{j,m,l} = 1.$$

We can now define a *biorthogonal wavelet basis*. The scaling basis function at the top-most level and the wavelet basis functions across all levels form a biorthogonal wavelet basis

$$\Psi = \{\varphi_{0,0}, \psi_{j,m} \mid j \in \mathcal{J}, m \in \mathcal{M}(j)\}$$

if the filter coefficients associated with the basis functions provide perfect reconstruction. A function $f \in L_2$ can then be represented as

$$f = \sum_{i \in \mathcal{I}} \left\langle f, \tilde{\psi}_i \right\rangle \psi_i = \sum_{i \in \mathcal{I}} \gamma_i\, \psi_i.$$

The γ_i are the basis function coefficients and $\mathcal{I}$ is an index set defined over all basis functions, including the scaling function at the top-most level.

Computing inner products to determine the basis function coefficients γ_i is computationally expensive and can be difficult, in particular if X is not a Cartesian space. The fast wavelet transform allows to project a signal into its basis representation and to reconstruct it in linear time $\mathcal{O}(n)$ with n being the size of the signal. An *analysis step* of the fast wavelet transform takes the form

$$\lambda_{j,k} = \sum_l \tilde{h}_{j,k,l}\,\lambda_{j+1,l} \quad \text{and} \quad \gamma_{j,m} = \sum_l \tilde{g}_{j,m,l}\,\lambda_{j+1,l},$$

and computes the basis function coefficients at level j from the scaling function coefficients at level $j+1$. A synthesis step takes the form

$$\lambda_{j+1,l} = \sum_k h_{j,k,l}\,\lambda_{j,k} + \sum_m g_{j,m,l}\,\gamma_{j,m}$$

and computes the scaling function coefficients at level $j+1$ from the basis function coefficients at level j.

Next to biorthogonal wavelet bases, also *semi-orthogonal* and *orthogonal* wavelets exist. For a semi-orthogonal wavelet basis, the spaces V_j and $\tilde{V}_j$ coincide, or equivalently,

$$\langle \varphi_{j,k}, \psi_{j,m} \rangle = 0 \quad \text{and} \quad \left\langle \tilde{\varphi}_{j,k}, \tilde{\psi}_{j,m} \right\rangle = 0.$$

If additionally the wavelet basis functions are orthogonal, that is

$$\langle \psi_{j,m}, \psi_{j',m'} \rangle = \delta_{j,j'}\delta_{m,m'}$$

then the wavelet basis is orthogonal. In this case the primary and dual basis functions coincide [12]. The ability to develop bases that are not fully orthogonal but which still permit to efficiently determine the basis function coefficients, via the inner product with the dual basis functions or the fast wavelet transform, adds significant flexibility to the design of bases and is one of reasons for the efficiency of wavelets in many different domains.

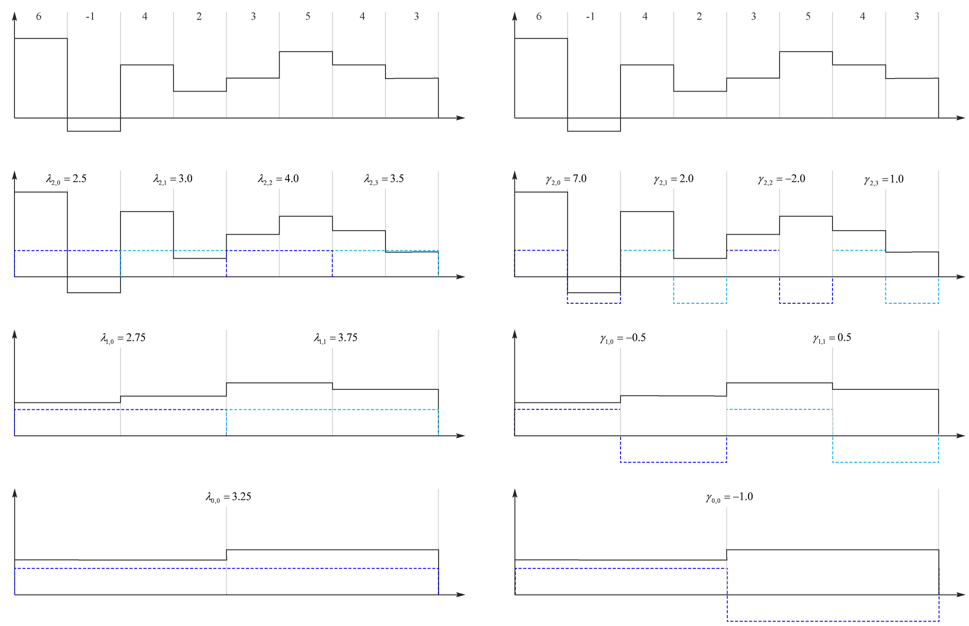

Figure 2: Wavelet transform for a discrete signal of length 8 with the Haar wavelet. Next to the signal (black) also the basis functions on each level are shown (blue and cyan). On the left and right side, respectively, the computations of the scaling function coefficients $\lambda_{j,k}$ and the wavelet basis function coefficients $\gamma_{j,m}$ are shown.

3 Wavelets in Action

After this rather mathematical treatment of wavelets let us now look at an example and revisit some of the properties of wavelet bases in a more concrete setting[8].

The wavelet basis we will employ is the (unnormalized) Haar basis [7]. The father and mother wavelets $\varphi(x)$ and $\psi(x)$, respectively, are shown in Figure 1, and the basis functions are formed as dyadic scales and translates of $\varphi(x)$ and $\psi(x)$

$$\varphi_{j,k} \equiv \varphi(2^j x - k) \quad \text{and} \quad \psi_{j,m} \equiv \psi(2^j x - m) \quad \text{with} \quad k, m = 0, \cdots, 2^j - 1.$$

The Haar basis is a first generation wavelet although it is easily possible to construct it in a second generation setting; see, for example, the thesis of Lessig for more details on how to construct Haar-like wavelets over arbitrary domains [9].

The (discrete) signal that is analysed in our example has length 8 and is shown in the top row of Figure 2. The left side of the figure shows how the scaling basis function coefficients $\lambda_{j,k}$ are computed. Note how the dimension of the spaces V_j is reduced by half when we go from level $j+1$ to level j. This is the realization of the nested structure of the spaces $V_j \subset V_{j+1}$. The difference spaces W_j are not shown explicitly in Figure 2 but these contain the information that is lost by downsampling V_{j+1} to obtain V_j.

The right column in Figure 2 shows that the magnitude of the wavelet basis function coefficients $\gamma_{j,m}$ is a measure for the local variation in a signal; coefficient $\gamma_{2,0}$, for example,

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[8] Due to the flexible nature of wavelets the following discussion does not apply to the same degree to all different wavelet bases.

has the largest magnitude and it is easy to see that the corresponding region of the signal is in fact those with the biggest variation. This property of the wavelet basis function coefficients holds not only for the Haar basis but can observed for all wavelet representations; the scaling functions act as average operator, or low frequency filter, whereas the wavelet basis functions are difference operators, or high-pass filter. The *correlation* that can be found in *all* natural signals – from images on YouTube to cosmological radiation – makes this one of the keys to the efficiency and versatility of wavelets. For data compression applications, for example, it can be exploited that most of the wavelet basis function coefficients for correlated signals are small and can thus be disregarded without significant error in the signals. For data analysis applications large basis function coefficients are of interest because deviations from typical behaviour (or correlations) is usually considered as "feature" (cf. [13]).

The basis function coefficient $\gamma_{2,0} = 7$, with its large magnitude, does not only tell us that there is a feature but it also tells us where the feature is. For the wavelet basis function coefficients the index m represents the space localization of the wavelet basis functions whereas the band or scale index j corresponds to the frequency localization. In our example the feature is thus at the beginning of the signal and it has a high frequency.

4 Further Readings

There are currently *thousands* of books on wavelets and even more papers with the same subject. Below is a short (and biased) list of literature the interested reader might consider as helpful for further explorations.

- *Wavelets in Computer Graphics* [15] The book provides a very gentle introduction with many (visual) examples from Computer Graphics.
- *Wavelet Primer* [14] The paper is a summary of the book *Wavelets in Computer Graphics* (see above).
- *The Lifting Scheme: A Custom-Design Construction of Biorthogonal Wavelets* [16]: The paper introduces the lifting schemes which is an important tool for second generation wavelets and which allows to construct more sophisticated wavelets from simpler ones.
- *SIGGRAPH 1996 Course Notes* [19]: Notes from a tutorial on second generation wavelets and the lifting scheme.
- *Spherical Wavelets* [12] The paper describes how wavelets on the sphere $\mathbb{S}^2$ can be constructed and it exemplifies the power of second generation wavelets.
- *SOHO Wavelets* [9] The master thesis provides a comprehensive introduction into second generation wavelet theory and explains how a basis with custom properties can be constructed.
- *A New Class of Unbalanced Haar Wavelets that form an Unconditional Basis for L_p on General Measure Spaces* [6] The paper provides a proof showing that second generation Haar-like wavelets are (unconditional) bases.
- *An Introduction to Frames and Riesz Basis* [2] The book provides the mathematical theory behind overcomplete representations, so-called frames.
- *Two Dimensional Wavelets and their Relatives* [1] The book approaches wavelets from a group-theoretic approach; the authors are theoretical physicists.
- *Ripples in Mathematics* [8] A standard introductory book on wavelets.
- *Wavelets: What's next?* [18] The essay describes (in an amusing way) the past, present, and future of research in wavelets.

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