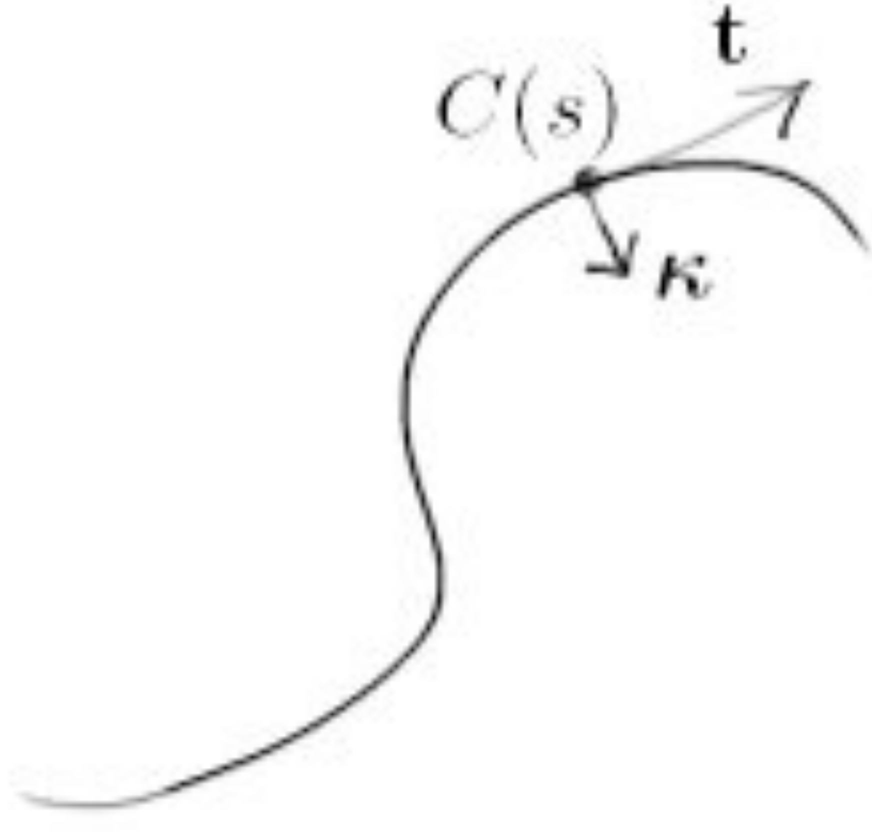
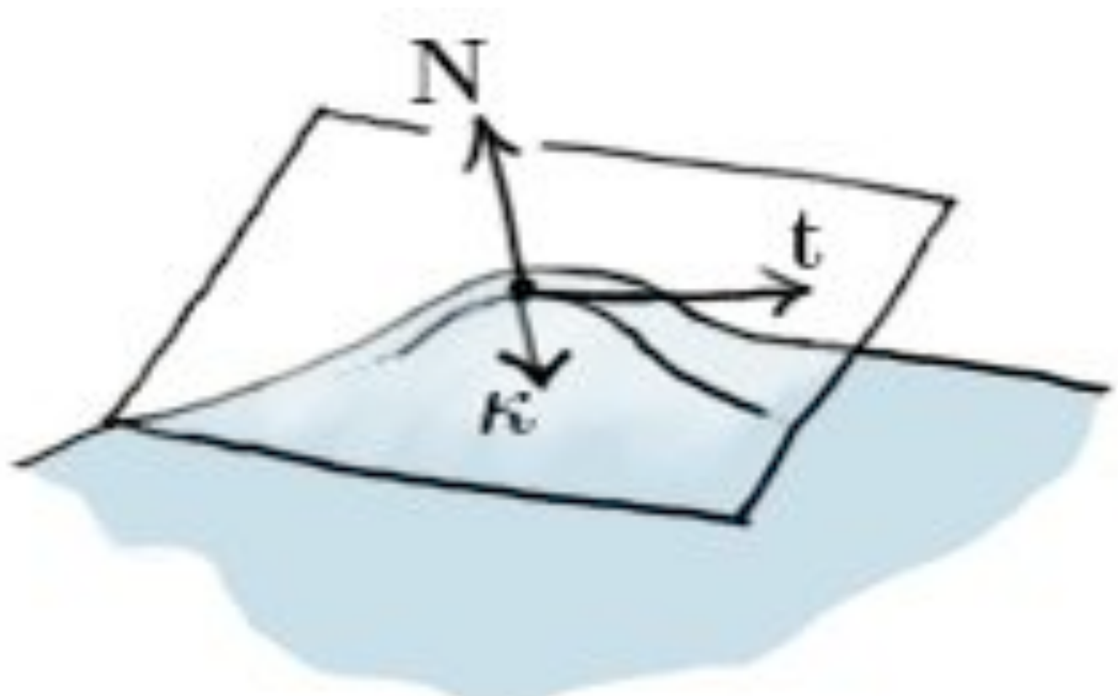
Discrete Differential Geometry

for triangle meshes
JAB 2013

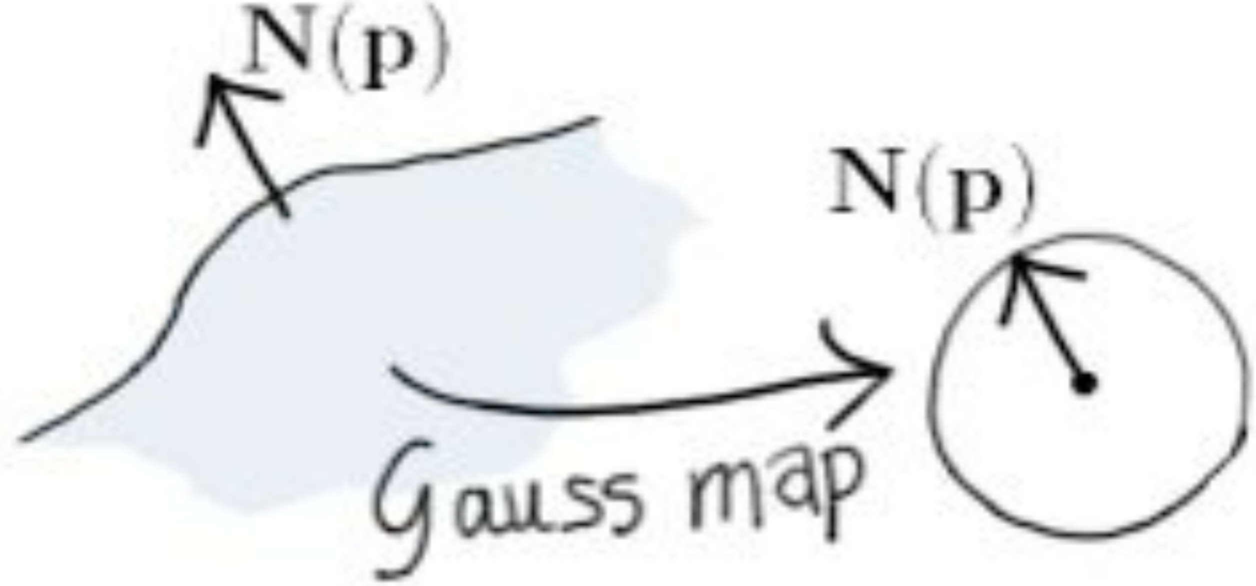
Consider a plane curve



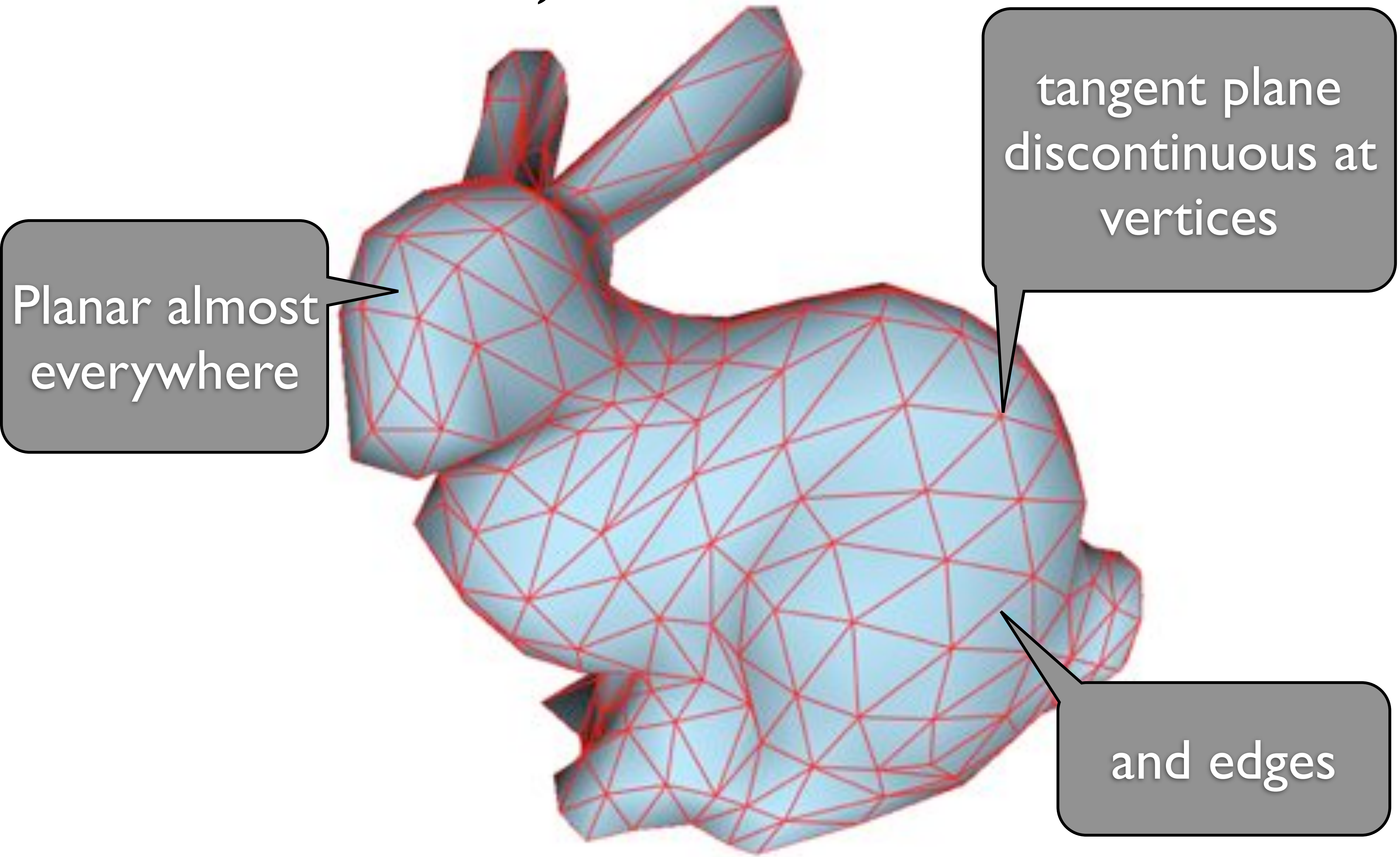
and a surface



... and the Gauss Map



Now, take a mesh

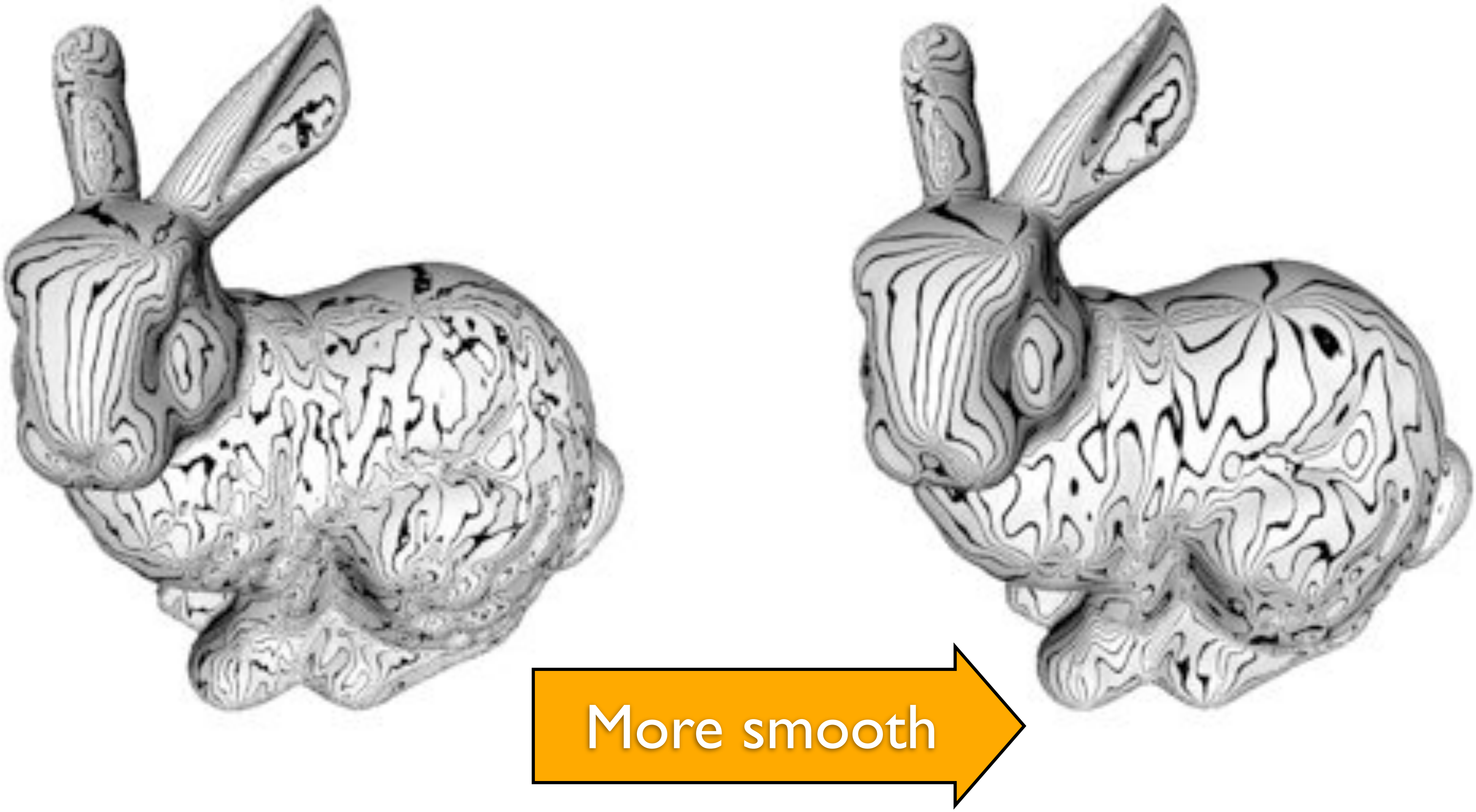


Making sense of curvature on a mesh

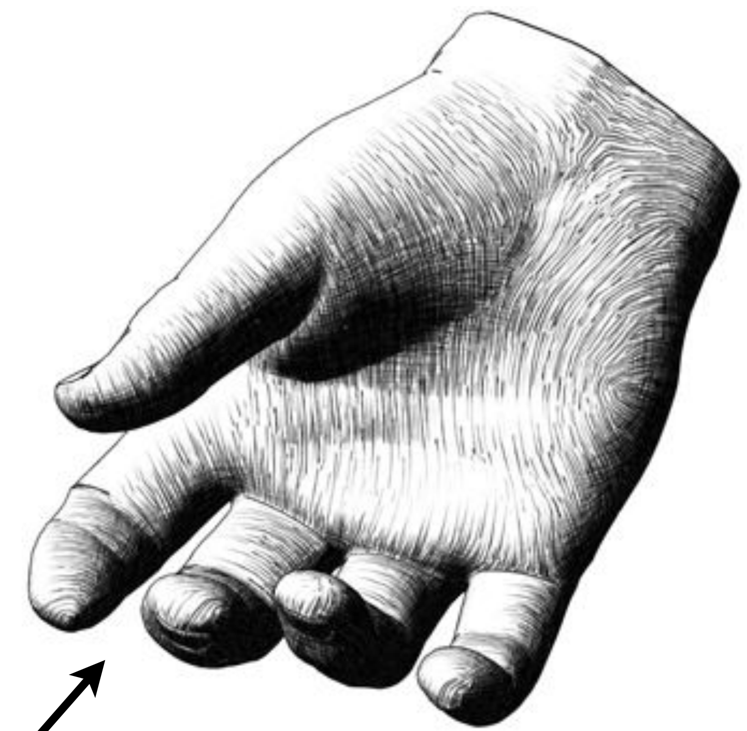
- At any point on a mesh, normal curvature in any direction is either infinite or 0
- However, we can fit a smooth surface to the mesh
- We can replace edges and corners with blends.
- We want curvature integrated over a region

Seeing Curvature

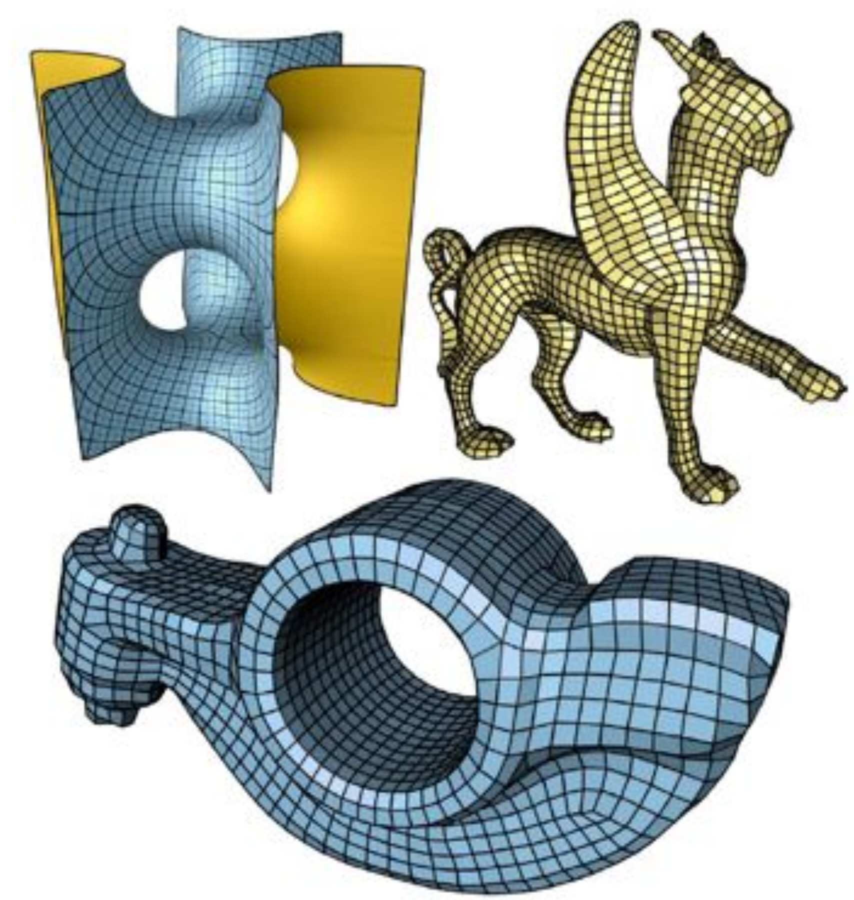
- Actually ... use reflection lines



Why compute curvature?



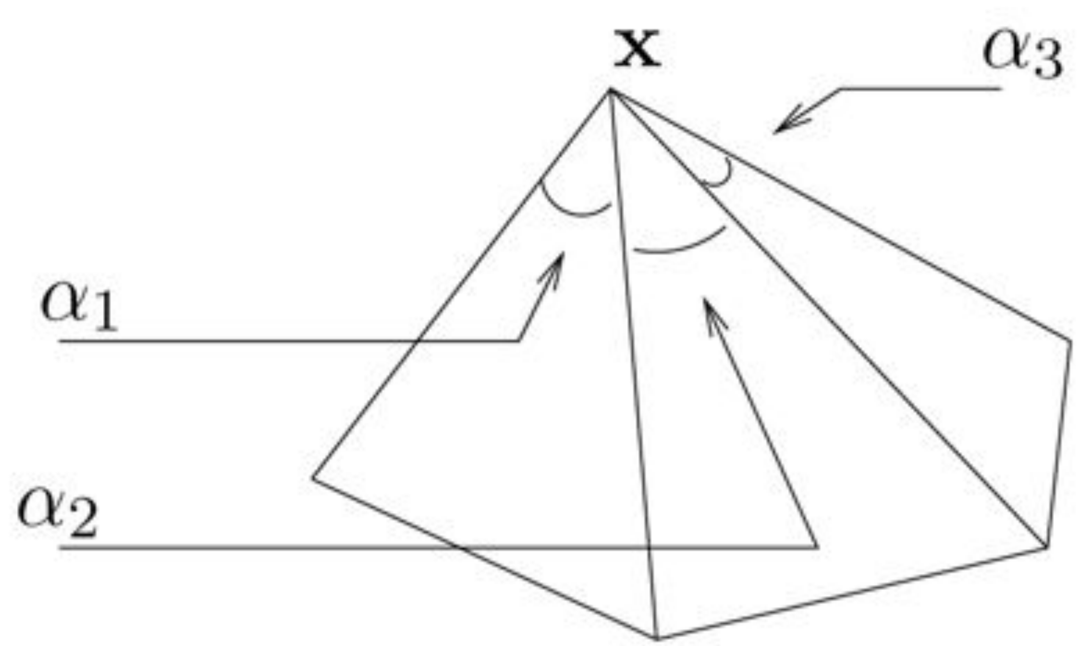
- Analysis
- non-photorealistic rendering
- remeshing
- improving geometry



Computing the Normal

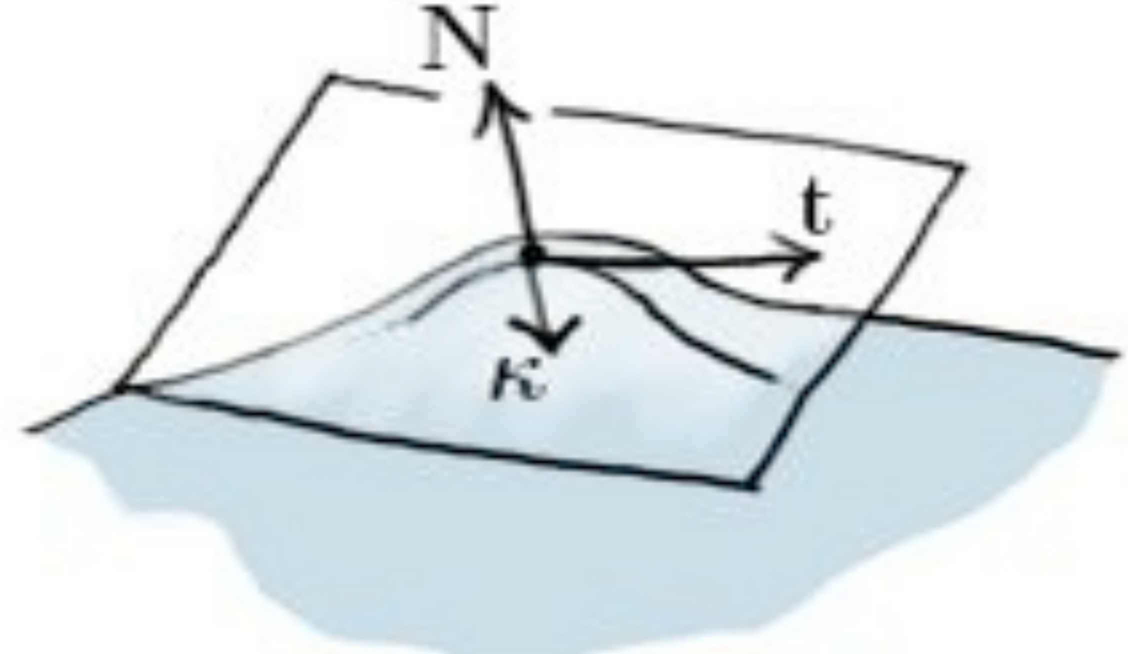
The angle weighted normal,

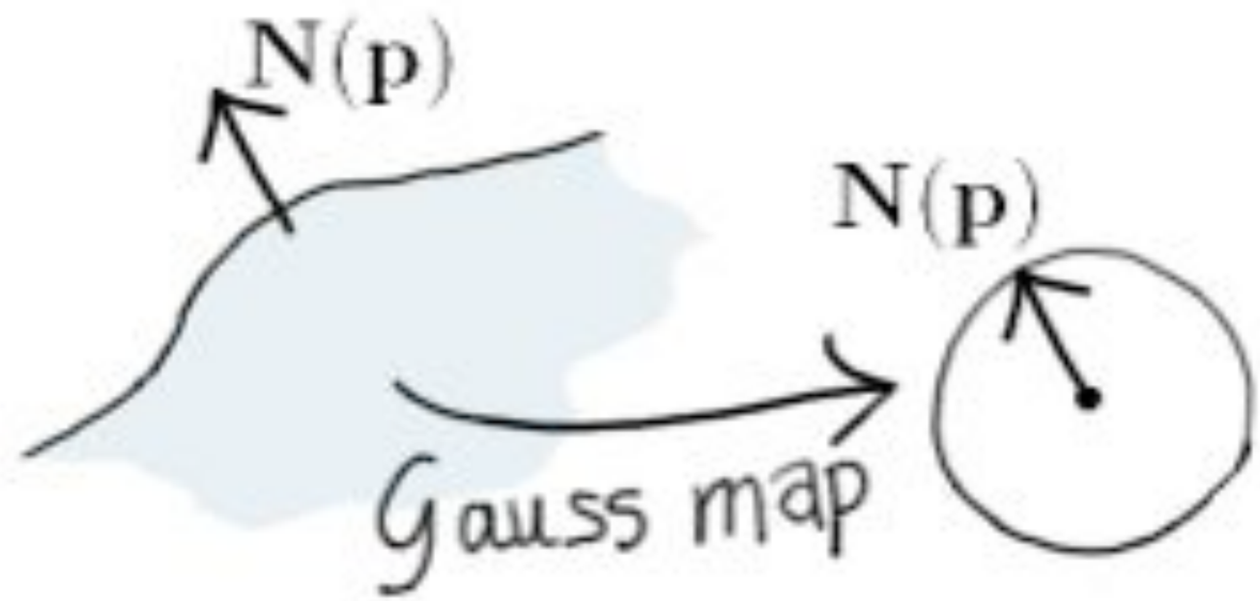
$$\mathbf{n}_\alpha = \frac{\sum_i \alpha_i \mathbf{n}_i}{||\sum_i \alpha_i \mathbf{n}_i||}$$



Computing Normal Curvature

- We need the shape operator:
  - Maps tangent plane directions to normal derivative
  - Derivative of the Gauss map
  - We need: Smooth quadratic patch fitted to surface

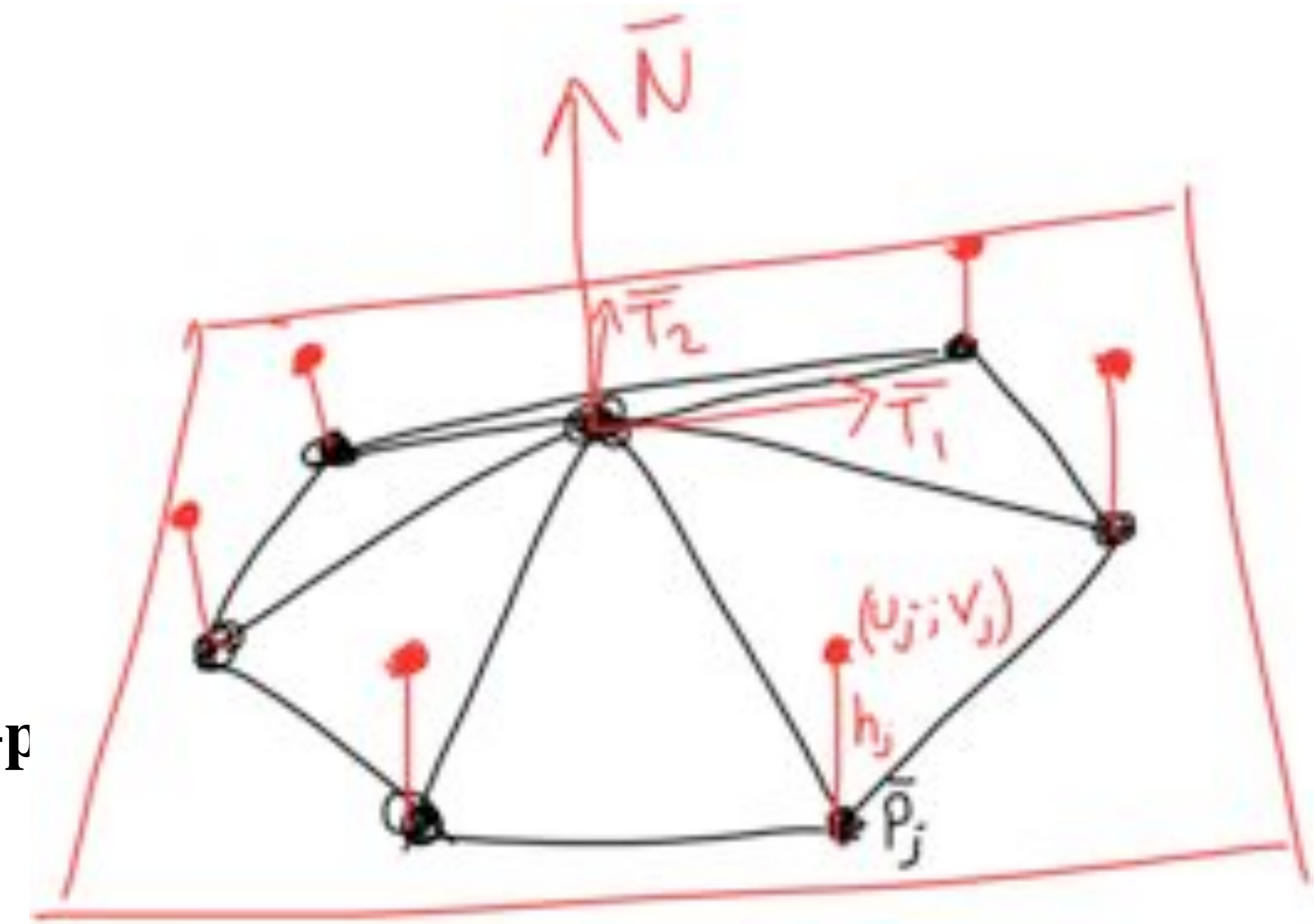




Fitting a Patch

$$f(u,v) = \frac{1}{2}(au^2 + 2buv + cv^2)$$

- For each vertex
- Compute frame ($\mathbf{N}$,$\mathbf{T_1}$,$\mathbf{T_2}$)
- Project 1-ring vertices: $(u_j,v_j)=\mathbf{T}^T(\mathbf{p}_j - \mathbf{p}_i)$
- Get height values: $h_j = \mathbf{N}\cdot(\mathbf{p}_j - \mathbf{p}$
- Now fit (on next slide)



Fitting a patch

Coefficients of f: a, b, and c are found by solving:

$$\begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \frac{u_j^2}{2} & u_j v_j & \frac{v_j^2}{2} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \cdot \\ \cdot \\ h_j \\ \cdot \\ \cdot \end{bmatrix}$$

in the least squares sense

Shape Operator

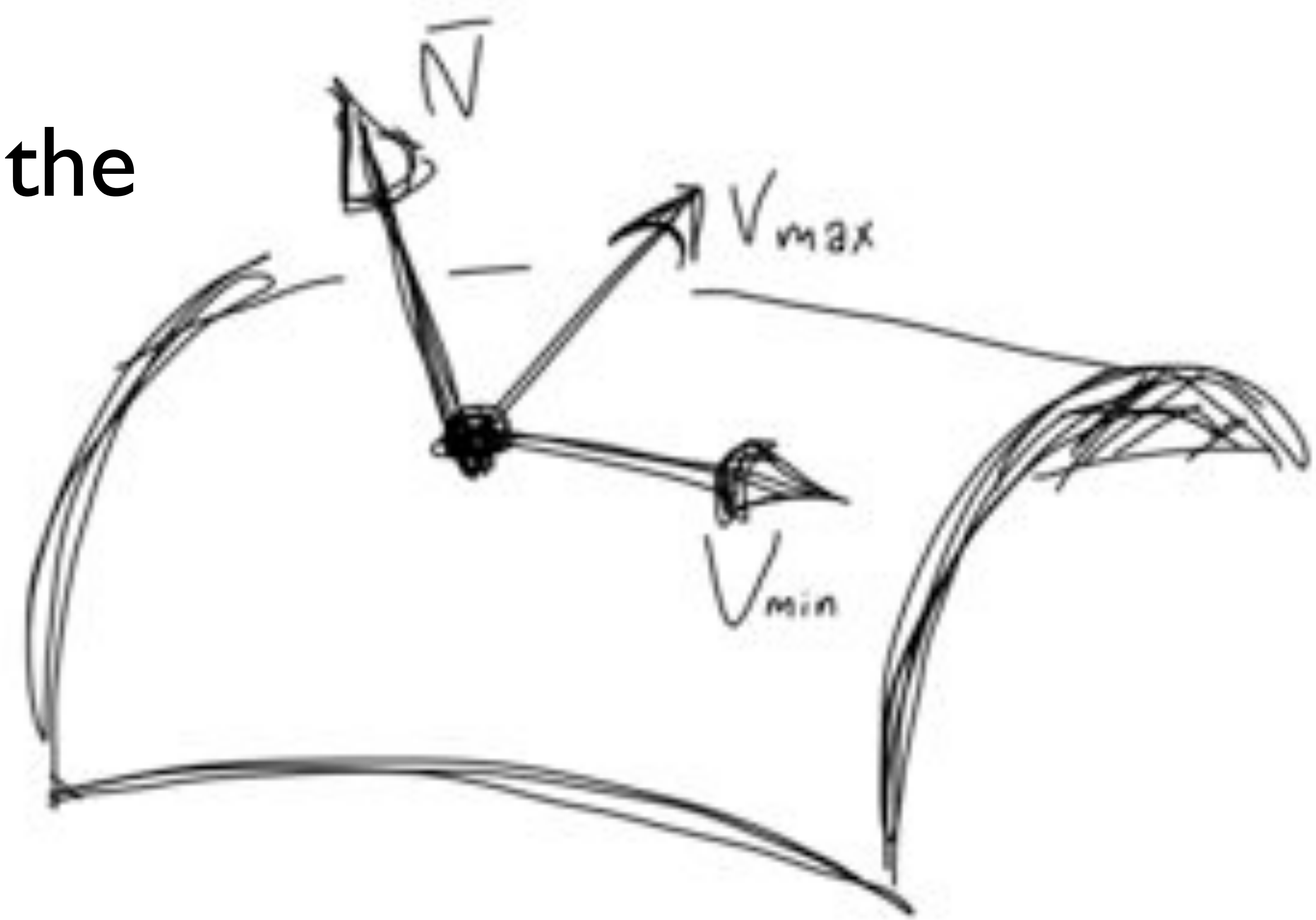
- Now, the shape operator is

$$\mathbf{S} = -\begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

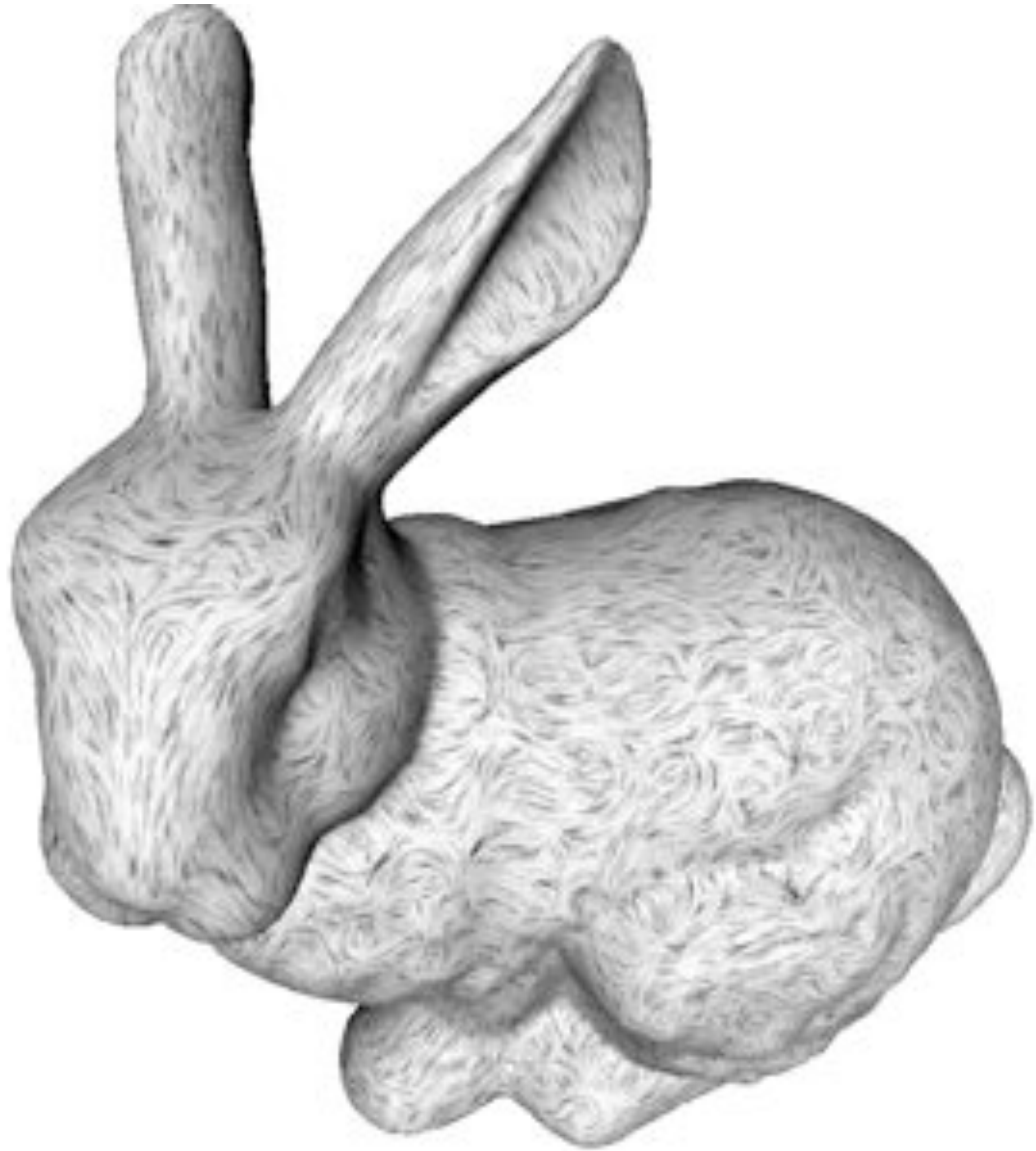
- since $f_u(0,0) = f_v(0,0) = 0$
- and *a,b* and *c* are 2. order derivatives of $f(u,v) = 0.5\ (au^2 + buv + cv^2)$

Principal Curvature

- Directions of min and max principal curvature v_{min} and v_{max} are eigenvectors of **S**
- κ_{min} and κ_{max} are the eigenvalues of **S**



Curvature Lines



Ridge Detection

- curvature extrema along max curvature direction
- Surprisingly hard to compute in a stable way

Gaussian and Mean Curvature

- The Gaussian curvature
  $K = \kappa_{\min} \kappa_{\max}$

Gauss Map area to surf area

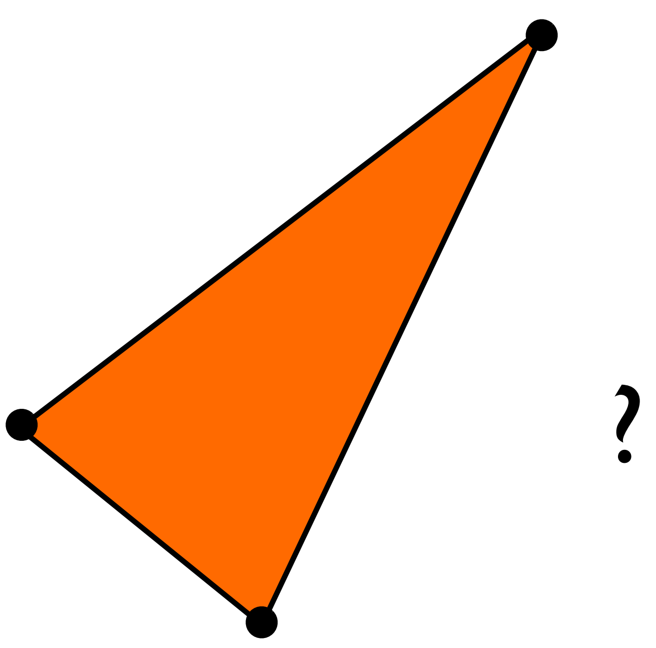
- The mean curvature
  $H = \kappa_{\min} + \kappa_{\max}$

Length of area gradient at a point

- Both of these have far more meaningful definitions

Euler-Poincare

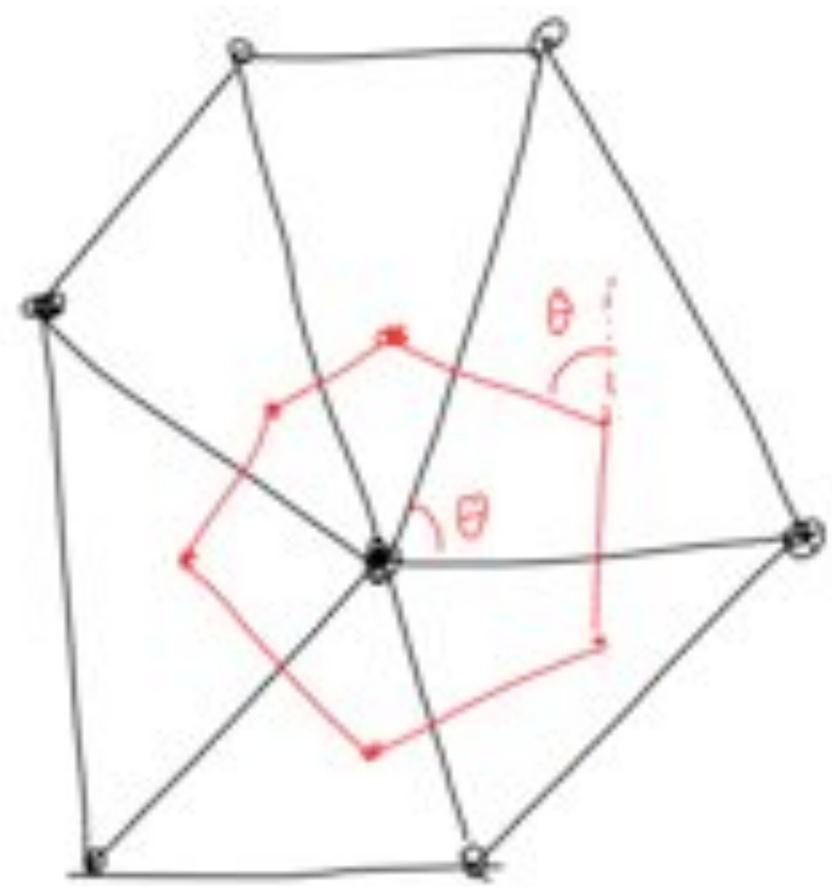
- Remember the Euler-Poincare formula
  $\#V - \#E + \#F = \chi$



Gauss-Bonnet

$$\int_R K \, \mathrm{d}A + \int_C \kappa_g \, \mathrm{d}s + \sum \phi_i = 2\pi\chi(R)$$

- Thus, given a path around the vertex (with zero geodesic, we can compute ...



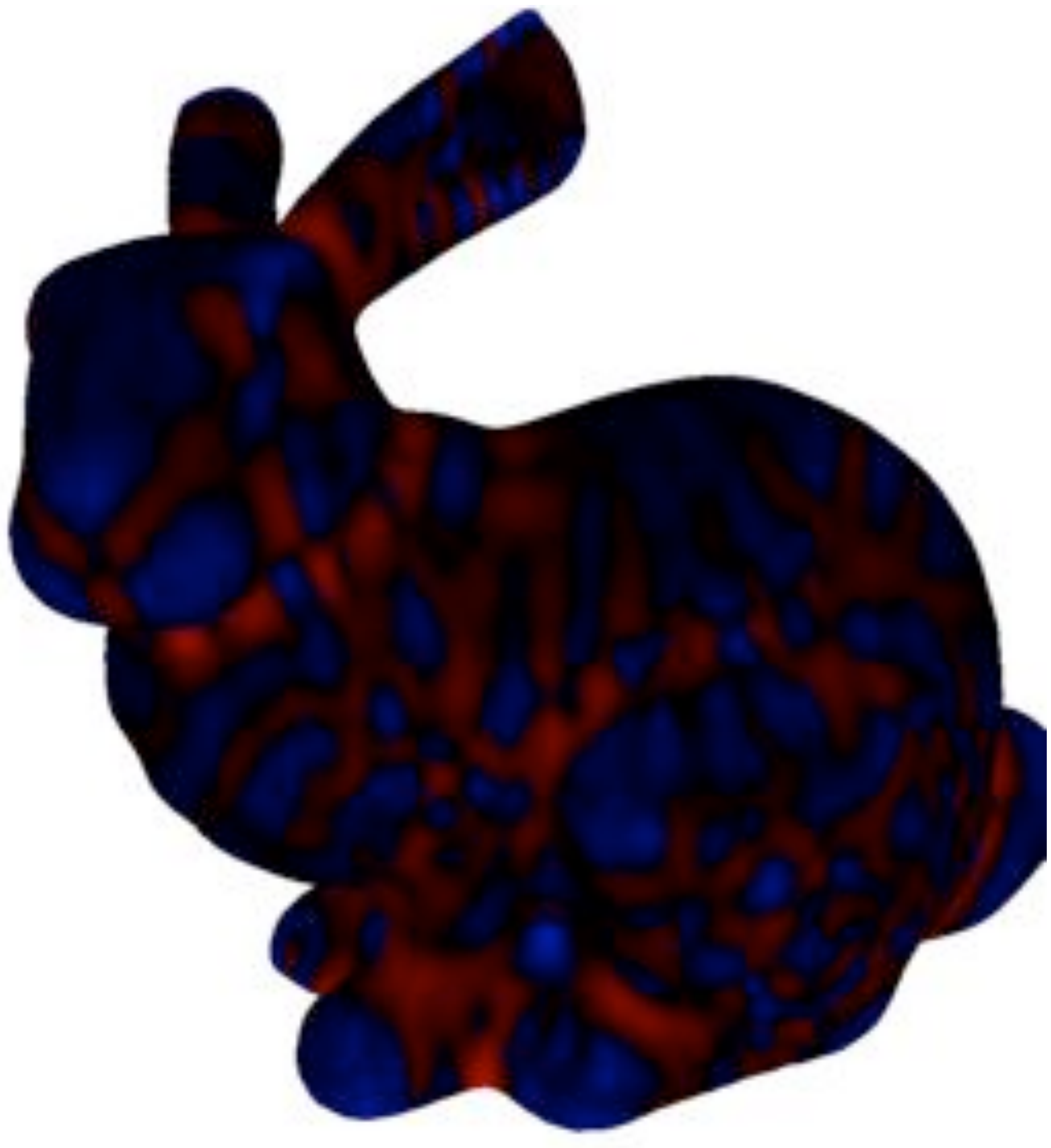
Gaussian Curvature

$$K(\mathbf{p}_i) = \frac{2\pi - \sum_j \theta_j}{\frac{1}{3}\sum_{\mathbf{p}_i \in t_j} A_j}$$

- Note, we could also start from a definition of Gaussian curvature:

$$K = \lim_{A_S \to 0} \frac{A_G}{A_S}$$

Gaussian Curvature

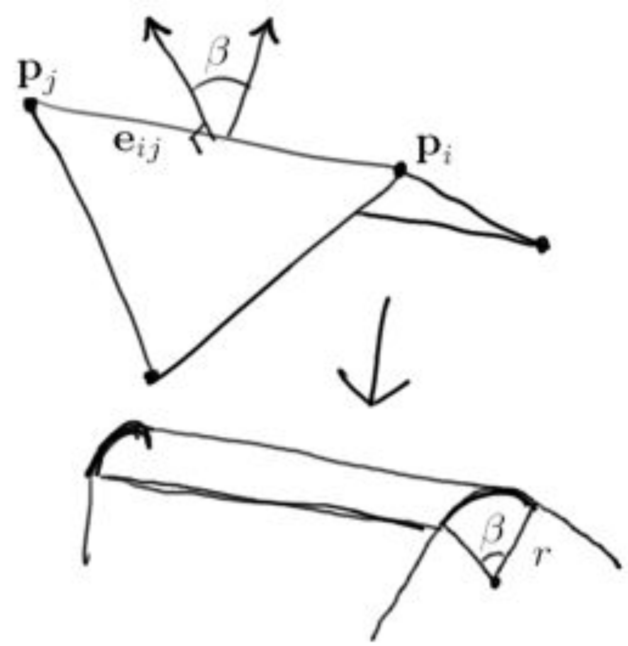


Integral of Gaussian Curvature

- For a closed surface, it is really easy:

$$\int_S K = \sum_{i=1}^{\nu} \left[2\pi - \sum_j \theta_j \right]$$

Mean Curvature on an edge



If we replace an edge **e** of dihedral angle β with a blend

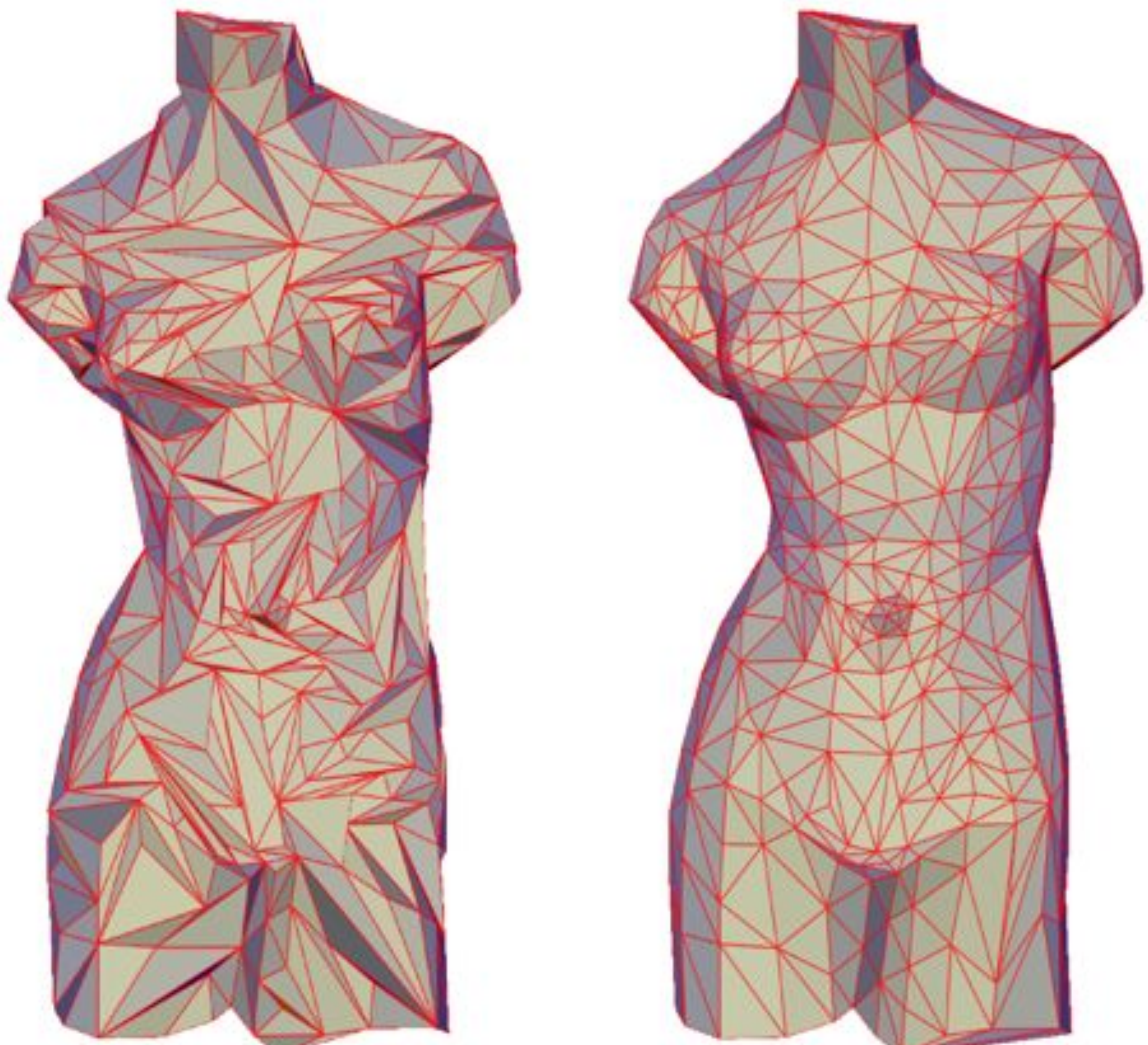
$$\int_B H = \frac{1}{2}(0 + \frac{1}{r})(r\beta\|\mathbf{e}\|) = \frac{1}{2}\beta\|\mathbf{e}\|$$

independent of radius r !!

Integral Abs Mean Curvature

- The IAMC is $$\int_S |H| = \frac{1}{2}\sum_{i=1}^{|\mathcal{E}|} |\beta_i| \|\mathbf{e}_i\| \quad .$$

Minimizing IAMC



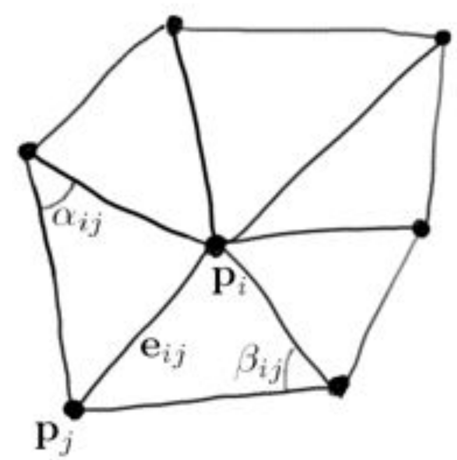
Return to the Normal

- A surface is minimal it its area is minimal given its boundary
- If so, $H = 0$
- Thus H relates to the area gradient. In fact

$$2\mathbf{H} = \lim_{A \to 0} \frac{\nabla A}{A}$$

Mean Curvature Normal

in a mesh, use the 1-ring area, $A_i^{1-\text{ring}} = \sum_{\mathbf{p}_i \in t_j} A_{t_j}$



Thus:

$$\mathbf{H}(\mathbf{p}_i) = \frac{1}{2}\frac{\nabla A_i^{1-\text{ring}}}{A_i^{1-\text{ring}}} = \frac{1}{4A_i^{1-\text{ring}}} \sum_{\mathbf{p}_j \in \mathcal{N}_i} (\cot \alpha_{ij} + cot \beta_{ij})(\mathbf{p}_i - \mathbf{p}_j) \quad ,$$

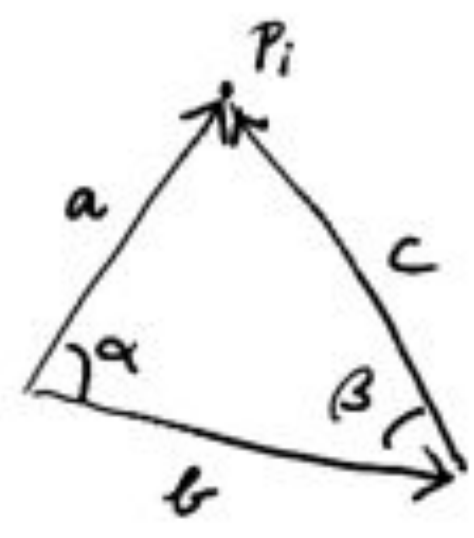
A spatial average, but where do the cots come from?

Triangle Area Gradient

The gradient of the vertex area is equal to the sum:

$$\nabla A_i^{1-\mathrm{ring}} = \sum_{\mathbf{p}_i \in t_j} \nabla A_{t_j}(\mathbf{p}_i)$$

where $A_{t_j}(\mathbf{p}_i) = \frac{\|\mathbf{b}\times\mathbf{a}\|}{2}$ and the terms are as shown:



Rewriting............

$$
\begin{aligned}
\nabla A_{t_j}(\mathbf{p}_i) &= \frac{(\mathbf{b}\times\mathbf{a})\times\mathbf{b}}{2\|\mathbf{b}\times\mathbf{a}\|} \\
&= \frac{(\mathbf{b}^t\mathbf{b})\mathbf{a}-(\mathbf{b}^t\mathbf{a})\mathbf{b}}{2\|\mathbf{b}\times\mathbf{a}\|} \\
&= \frac{(\mathbf{b}^t\mathbf{b})\mathbf{a}-(\mathbf{b}^t\mathbf{a})\mathbf{a}+(\mathbf{b}^t\mathbf{a})\mathbf{a}-(\mathbf{b}^t\mathbf{a})\mathbf{b}}{2\|\mathbf{b}\times\mathbf{a}\|} \\
&= \frac{-(\mathbf{c}^t\mathbf{b})\mathbf{a}}{2\|\mathbf{c}\times-\mathbf{b}\|}+\frac{(\mathbf{b}^t\mathbf{a})\mathbf{c}}{2\|\mathbf{b}\times\mathbf{a}\|} \\
&= \frac{(\mathbf{c}^t-\mathbf{b})\mathbf{a}}{2\|\mathbf{c}\times-\mathbf{b}\|}+\frac{(\mathbf{b}^t\mathbf{a})\mathbf{c}}{2\|\mathbf{b}\times\mathbf{a}\|} \\
&= \frac{1}{2}(\mathbf{a}\cot\beta+\mathbf{c}\cot\alpha)
\end{aligned}
$$

Mean Curvature

The Laplace-Beltrami Operator

- The mean curvature normal is also defined as the LBO

$$\Delta f = \nabla \cdot \nabla f$$

  applied to the vertex positions

- Not so mysterious

Litterature

Guide to Computational Geometry Processing, Bærentzen, J. Andreas ; Gravesen, Jens ; Anton, François ; Aanæs, Henrik, Springer 2012

Discrete Differential-Geometry Operators for Triangulated 2-Manifolds Mark Meyer, Mathieu Desbrun, Peter Schröder, and Alan H. Barr, VisMath 2002.

Software: GEL