CSC418 Computer Graphics

I'm not Professor Karan Singh

Course web site (includes course information sheet and discussion board):

`http://www.dgp.toronto.edu/~karan/courses/418/`

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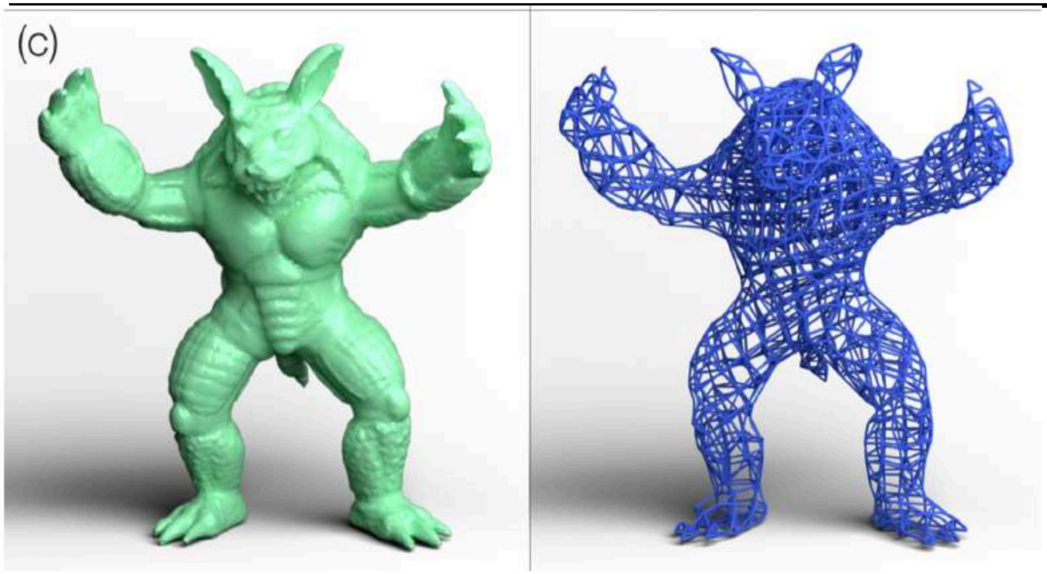
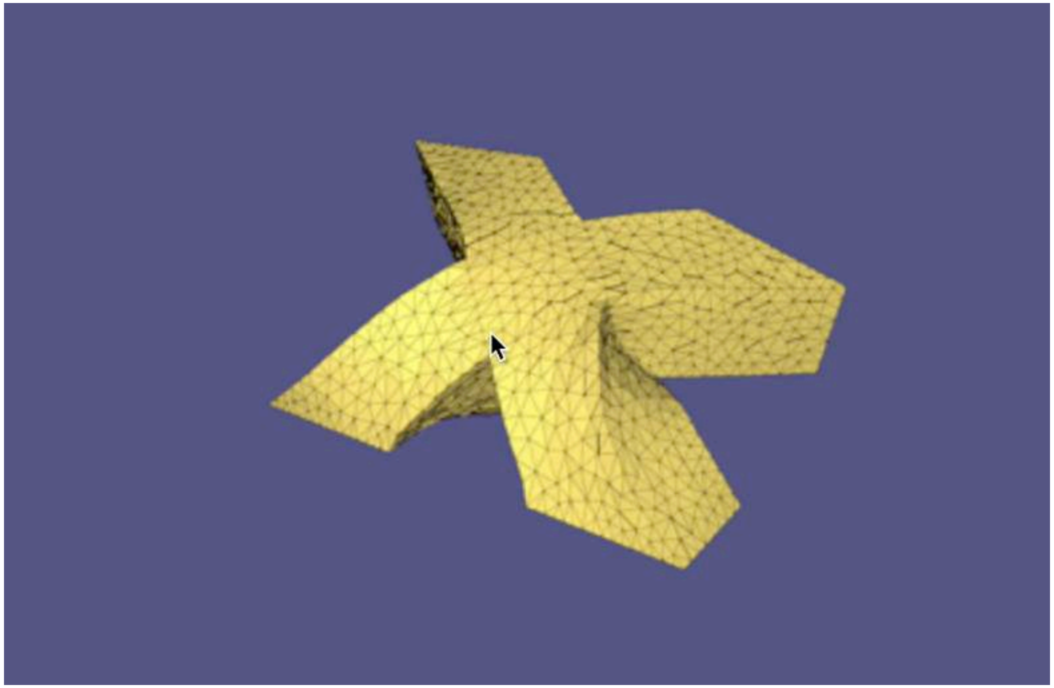
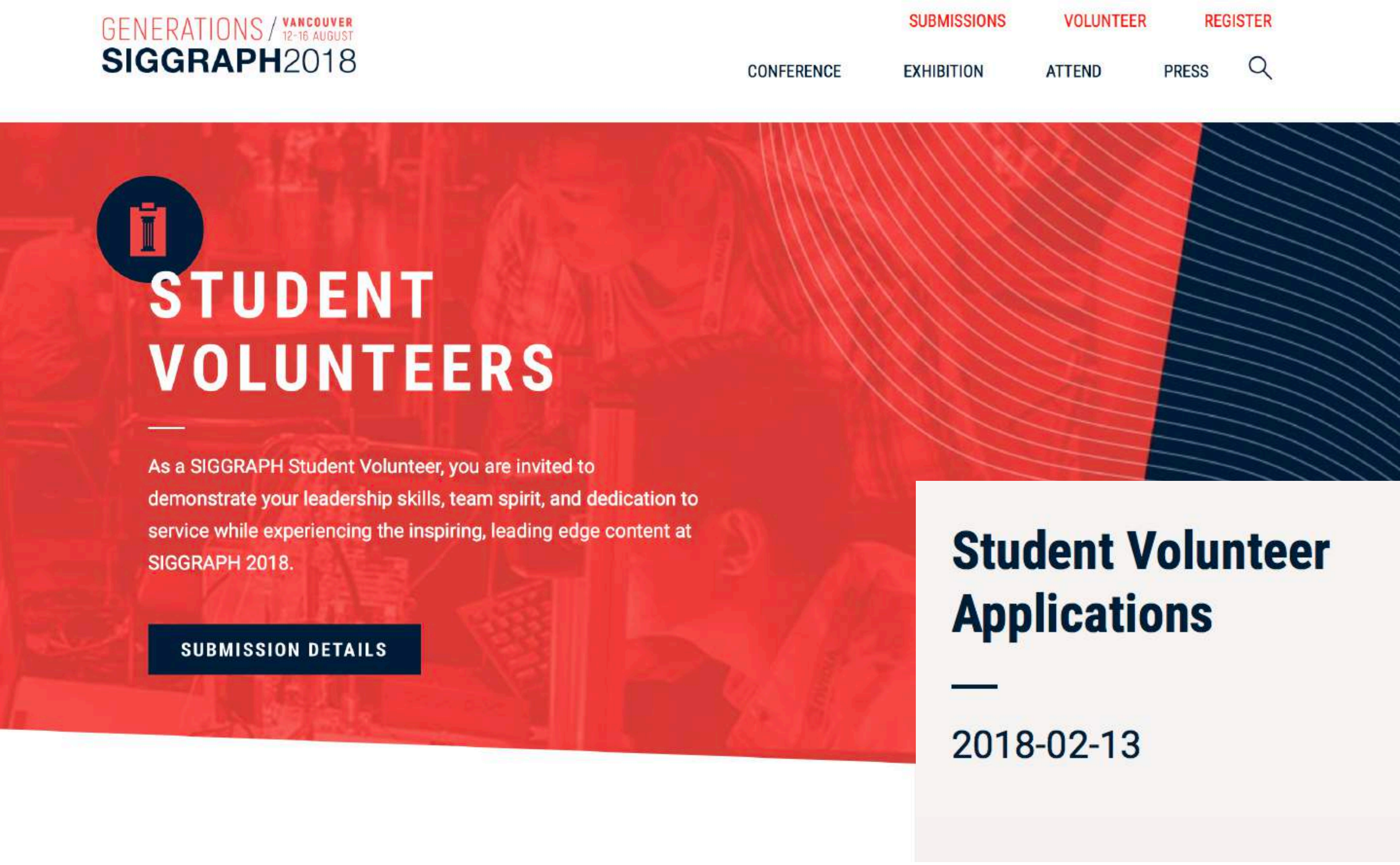
Textbooks: Fundamentals of Computer Graphics
OpenGL Programming Guide & Reference



3D Printable Structures

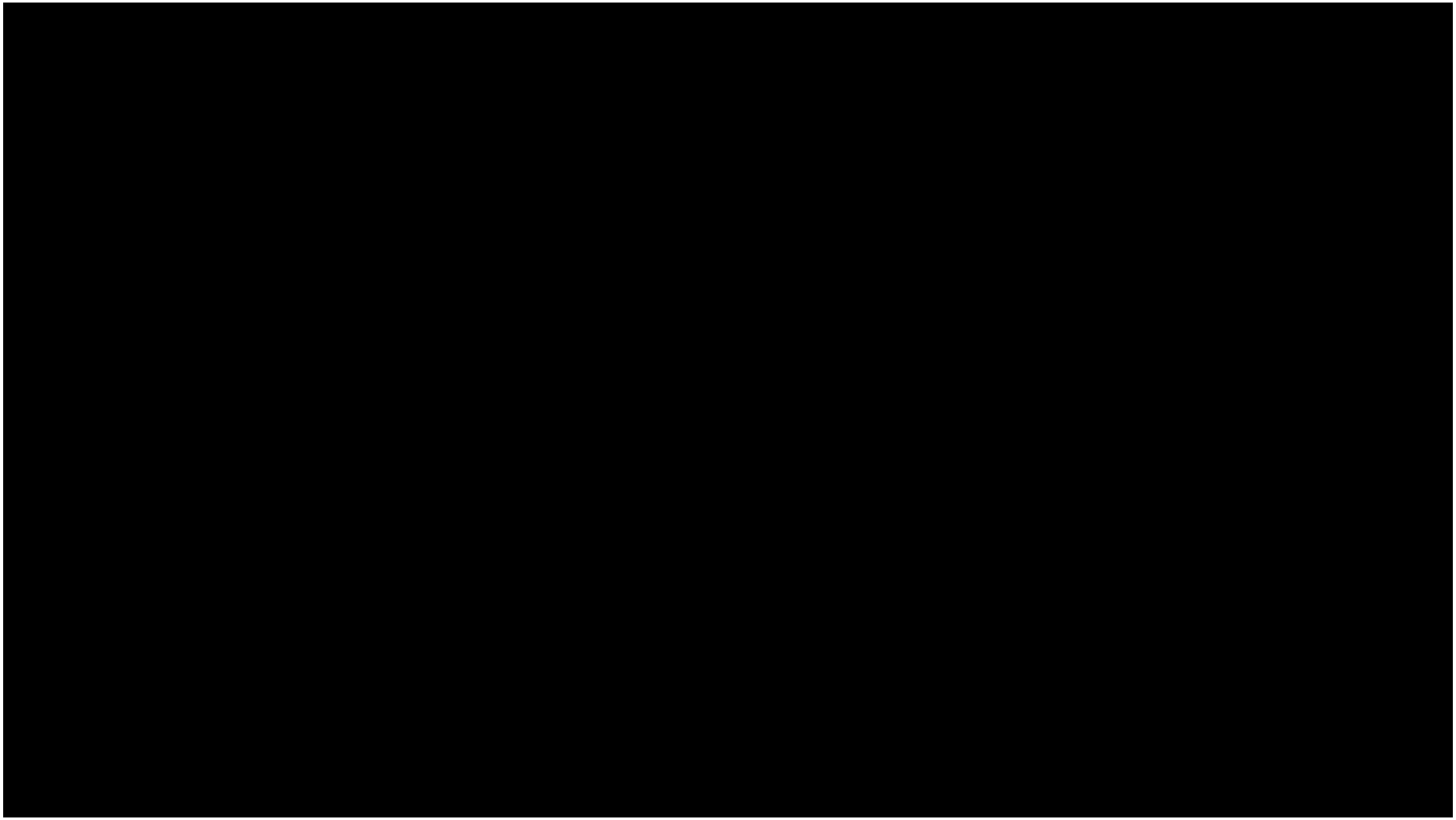


Real-time Physics using ML



https://s2018.siggraph.org/conference/conference-overview/student-volunteers/

Showtime:



Today's Topics

2. Review Implicit Curve Representation

3. Transformations in 2D

4. Coordinate-free geometry

5. 3D Objects (curves & surfaces)

6. Transformations in 3D

Questions about the Midterm

If you have a valid, **documented** reason for missing the midterm exam, your final exam will be worth 50%

Questions about the Assignment

Please contact the TAs via email at csc418tas@cs.toronto.edu

Topic 2.

2D Curve Representations

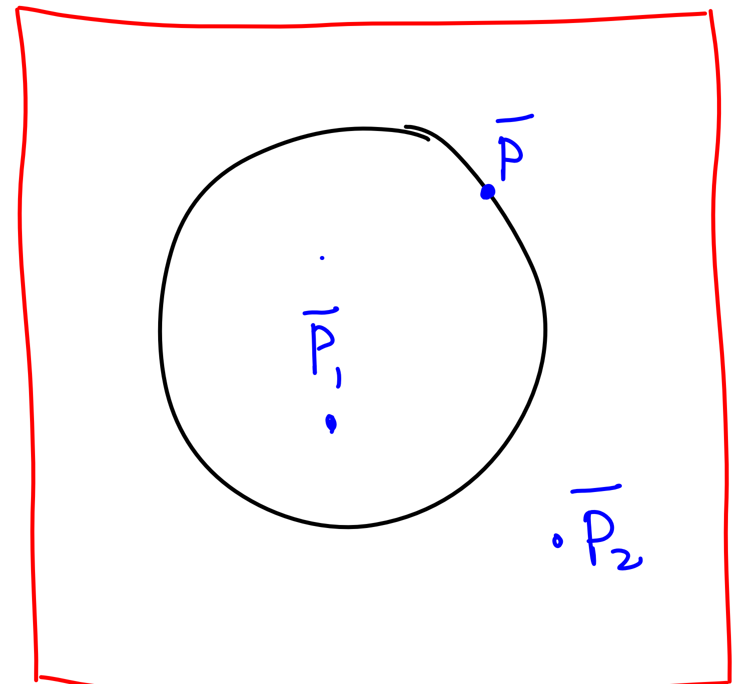
- Explicit representation
- Parametric representation
- Tangent & normal vectors
- Implicit representation

Implicit Curve Representation: Definition

A function $f(x,y)$ that is zero if and only if (x,y) is on the curve

$$f(\bar{p}) = 0$$

called the implicit equation of the curve



$$f(\bar{p}) = 0$$

$$f(\bar{p}_1) \neq 0$$

$$f(\bar{p}_2) \neq 0$$

Implicit Curve Representation: Definition

A function $f(x,y)$ that is zero if and only if (x,y) is on the curve

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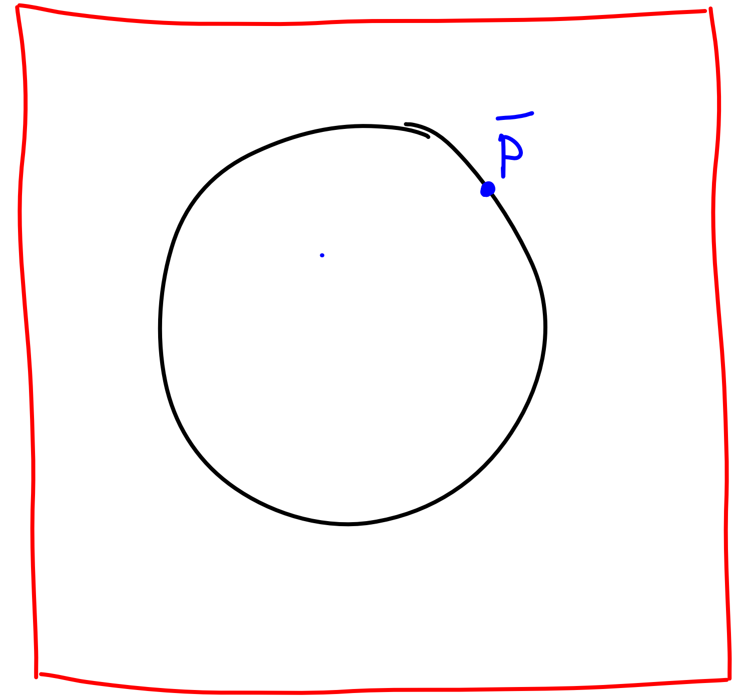


Circle with radius r centered at $(0,0)$:

$$f(x,y) = x^2 + y^2 - r^2$$

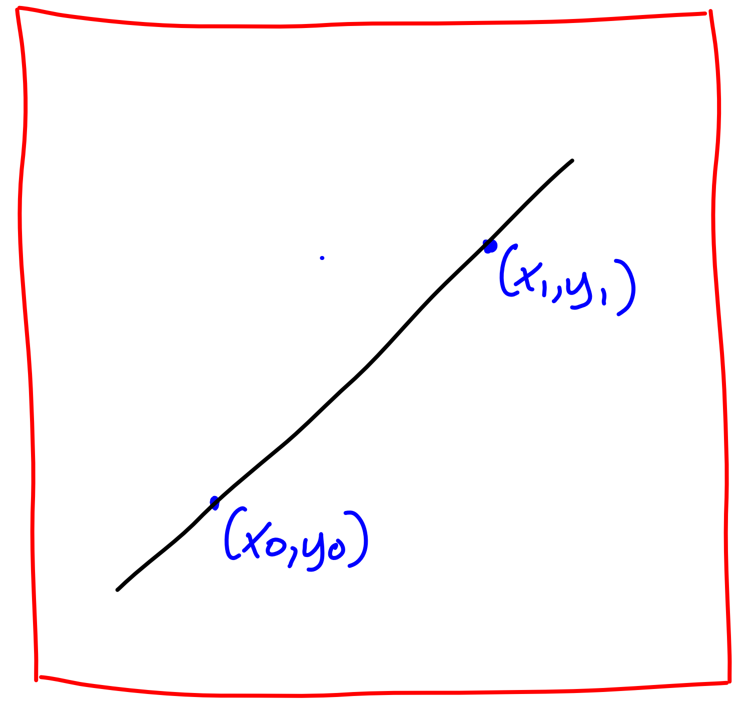
(because x, y must satisfy $x^2 + y^2 = r^2$)

Implicit Curve Representation: Definition

A function $f(x,y)$ that is zero if and only if (x,y) is on the curve

$$\boxed{f(\bar{p}) = 0}$$

called the implicit equation of the curve

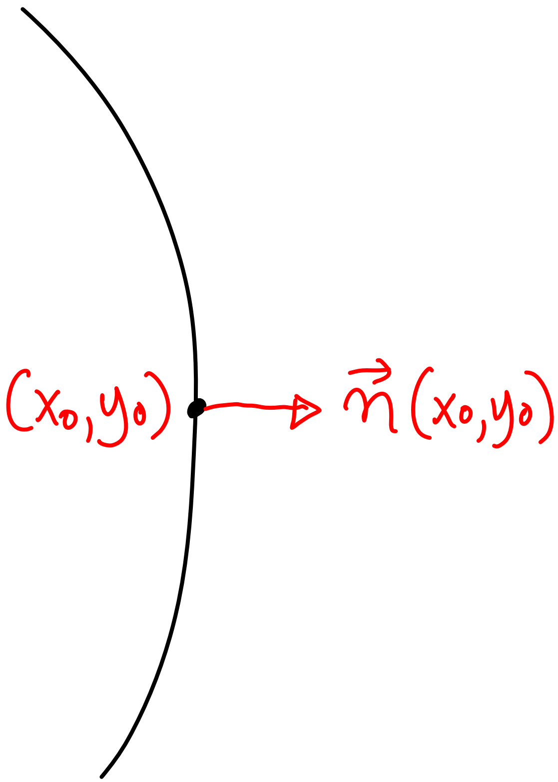


Line through (x_0, y_0) and (x_1, y_1)

$$f(x,y) = (y - y_0)(x_1 - x_0) - (y_1 - y_0)(x - x_0)$$

(because x, y must satisfy $\dfrac{y - y_0}{y_1 - y_0} = \dfrac{x - x_0}{x_1 - x_0}$)

Normal Vectors from the Implicit Equation



If $f(x,y)$ is the implicit eq of a curve and (x_0, y_0) is a point on it, then

$$\vec{n}(x_0, y_0) = \nabla f(x_0, y_0)$$

where $\nabla f(x_0, y_0)$ is the gradient of f at point (x_0, y_0), i.e.

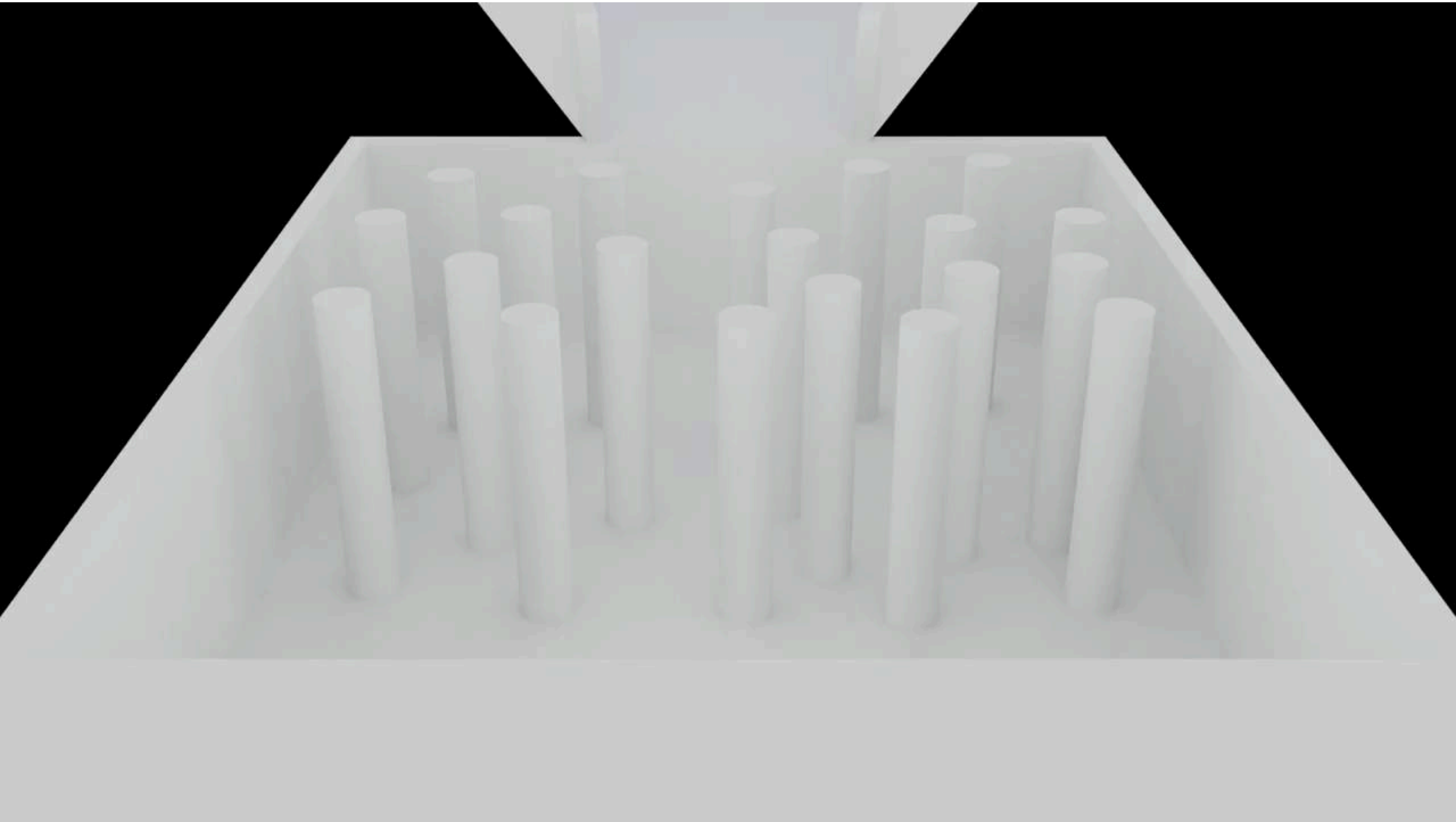
$$\nabla f(x_0, y_0) = \left(\frac{\partial f}{\partial x}(x_0), \frac{\partial f}{\partial y}(y_0) \right)$$

Topic 3:

2D Transformations

- Simple Transformations
- Homogeneous coordinates
- Homogeneous 2D transformations
- Affine transformations & restrictions

Transformations are Fun



Transformations

Transformation/Deformation in Graphics:

A function f, mapping points to points.
simple transformations are usually invertible.

$$[x\ y]^T \underset{f^{-1}}{\overset{f}{\rightleftarrows}} [x'\ y']^T$$

Applications:

- Placing objects in a scene.
- Composing an object from parts.
- Animating objects.

Processing Tree Demo!
https://processing.org/examples/tree.html

Lets start out simple...

Translate a point $[x\ y]^T$ by $[t_x\ t_y]^T$:

$$x' = x + t_x$$
$$y' = y + t_y$$

Rotate a point $[x\ y]^T$ by an angle t :

$$x' = x \cos t - y \sin t$$
$$y' = x \sin t + y \cos t$$

Scale a point $[x\ y]^T$ by a factor $[s_x\ s_y]^T$

$$x' = x\, s_x$$
$$y' = y\, s_y$$

Representing 2D transforms as a 2x2 matrix

Rotate a point $[x\ y]^T$ by an angle t :

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

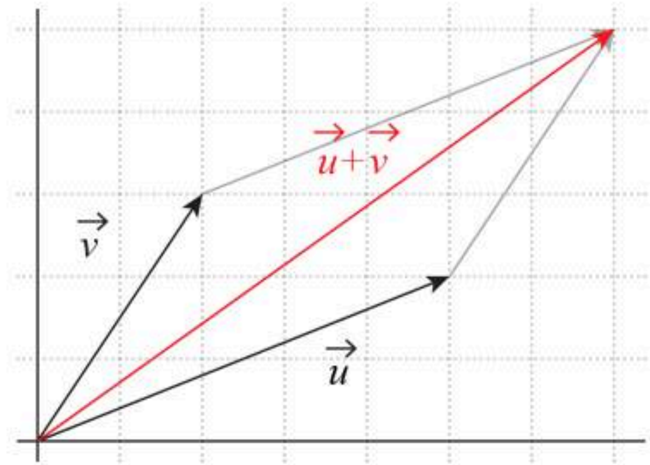
Scale a point $[x\ y]^T$ by a factor $[s_x\ s_y]^T$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

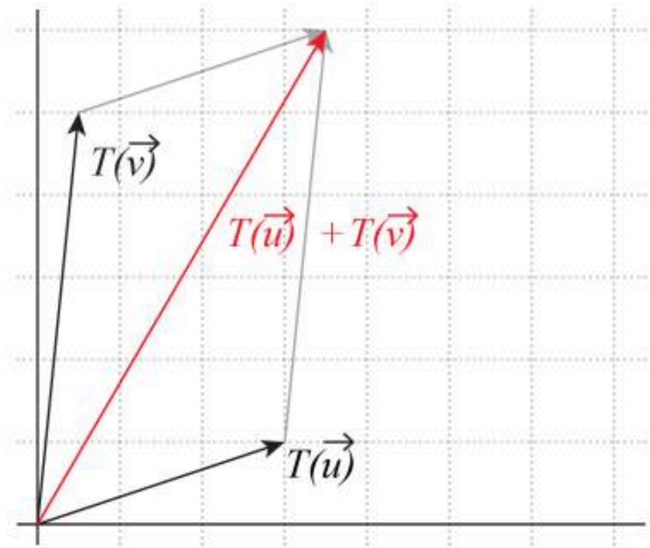
Linear Transformations

A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is called a linear transformation if it satisfies

1. $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$
   for all $\vec{u}, \vec{v} \in \mathbb{R}^n$.
2. $T(c\vec{v}) = cT(\vec{v})$
   for all $\vec{v} \in \mathbb{R}^n$ and all scalars c.



Linear transformations can be represented by *matrices*.



Remember how multiplication with a scalar is defined and that matrix multiplication is *distributive over addition*:

$A(B + C) = AB + AC$

Finding matrices

Remember: T is a linear transformation if and only if

$$T(c_1\vec{u} + c_2\vec{v}) = c_1 T(\vec{u}) + c_2 T(\vec{v})$$

Let's look at carthesian coordinates, where each vector $\vec{w}$ can be represented as a linear combination of the base vectors $\vec{b_1}$, $\vec{b_2}$:

$$\vec{w} = \begin{pmatrix} x \\ y \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

If we apply a linear transformation T to this vector, we get:

$$T(\begin{pmatrix} x \\ y \end{pmatrix}) = T(x \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \end{pmatrix}) = xT(\begin{pmatrix} 1 \\ 0 \end{pmatrix}) + yT(\begin{pmatrix} 0 \\ 1 \end{pmatrix})$$

Finding matrices

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$$= \begin{bmatrix} T\begin{pmatrix} 1 \\ 0 \end{pmatrix} & T\begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Transformation of a point is determined by a transformation of the basis vectors

Finding matrices

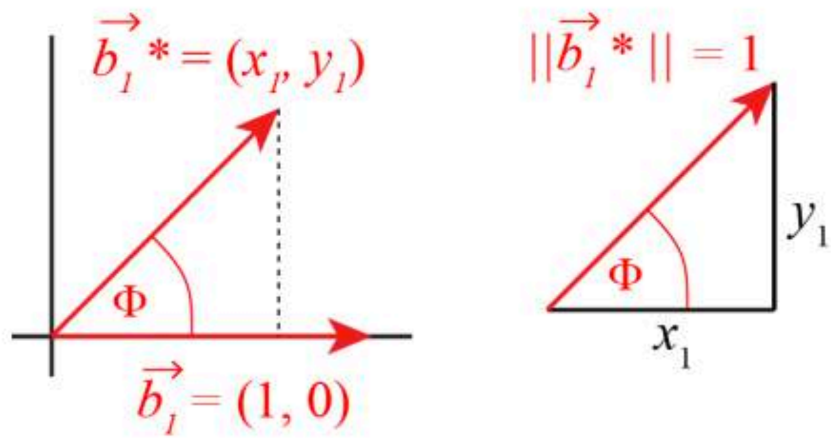
That gives us an easy method to find transformation matrices.

Example:

Counterclockwise rotation about an angle Φ

$$\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

First base vector $\vec{b_1}$ gives the first column:



$x_1 = \cos\Phi$ *and* $y_1 = \sin\Phi$

Second base vector $\vec{b_2}$
$\rightsquigarrow$ exercise

Representing 2D transforms as a 2x2 matrix

Rotate a point $[x\ y]^T$ by an angle t :

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Scale a point $[x\ y]^T$ by a factor $[s_x\ s_y]^T$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} s_x & 0 \\ 0 & s_y \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Translation ?

Representing 2D transforms as a 2x2 matrix

Rotate a point $[x\ y]^T$ by an angle t :

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Scale a point $[x\ y]^T$ by a factor $[s_x\ s_y]^T$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} s_x & 0 \\ 0 & s_y \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Translate a point $[x\ y]^T$ by $[t_x\ t_y]^T$:

$$x' = x + t_x$$
$$y' = y + t_y$$

Intuition via Shearing

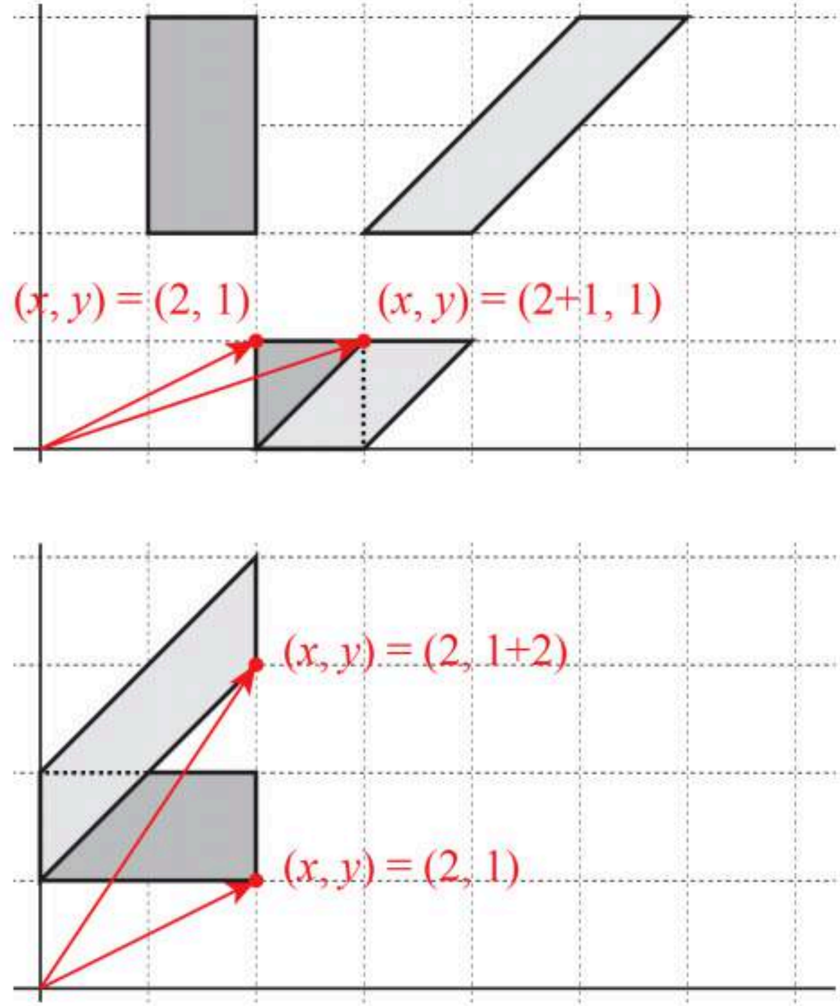
General case for shearing

... in X-direction:

$$\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + sy \\ y \end{pmatrix}$$

... in Y-direction:

$$\begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ sx + y \end{pmatrix}$$



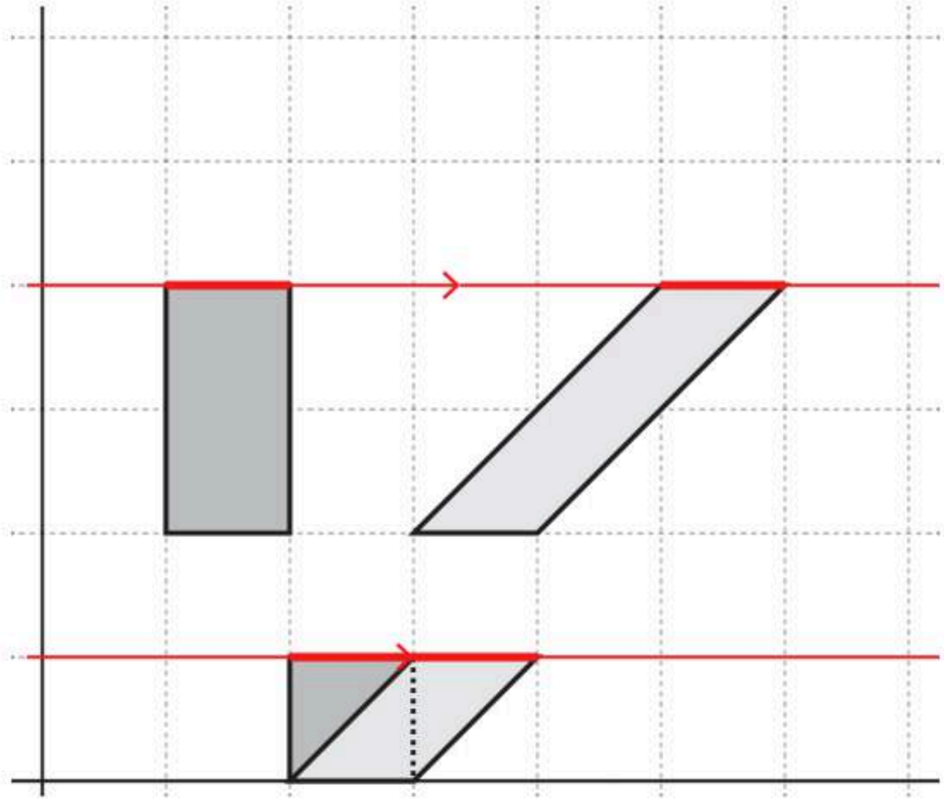
Translation via Shearing

Observation:

In 2D, shearing "pushes things sideways" (in $X-$direction) in a "fixed level" (the $Y-$value).

That "level" is a 1D subspace, i.e. a line.

Ergo, we are doing a translation (in 1D) using matrix multiplication (in 2D).

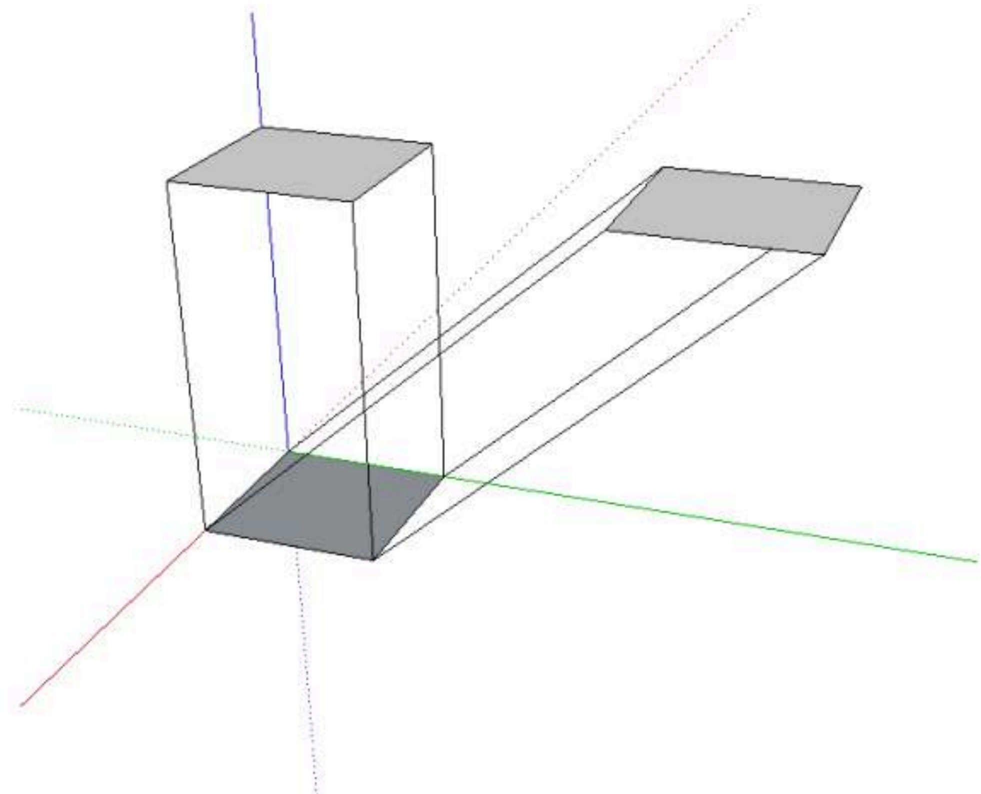


Homogeneous coordinates

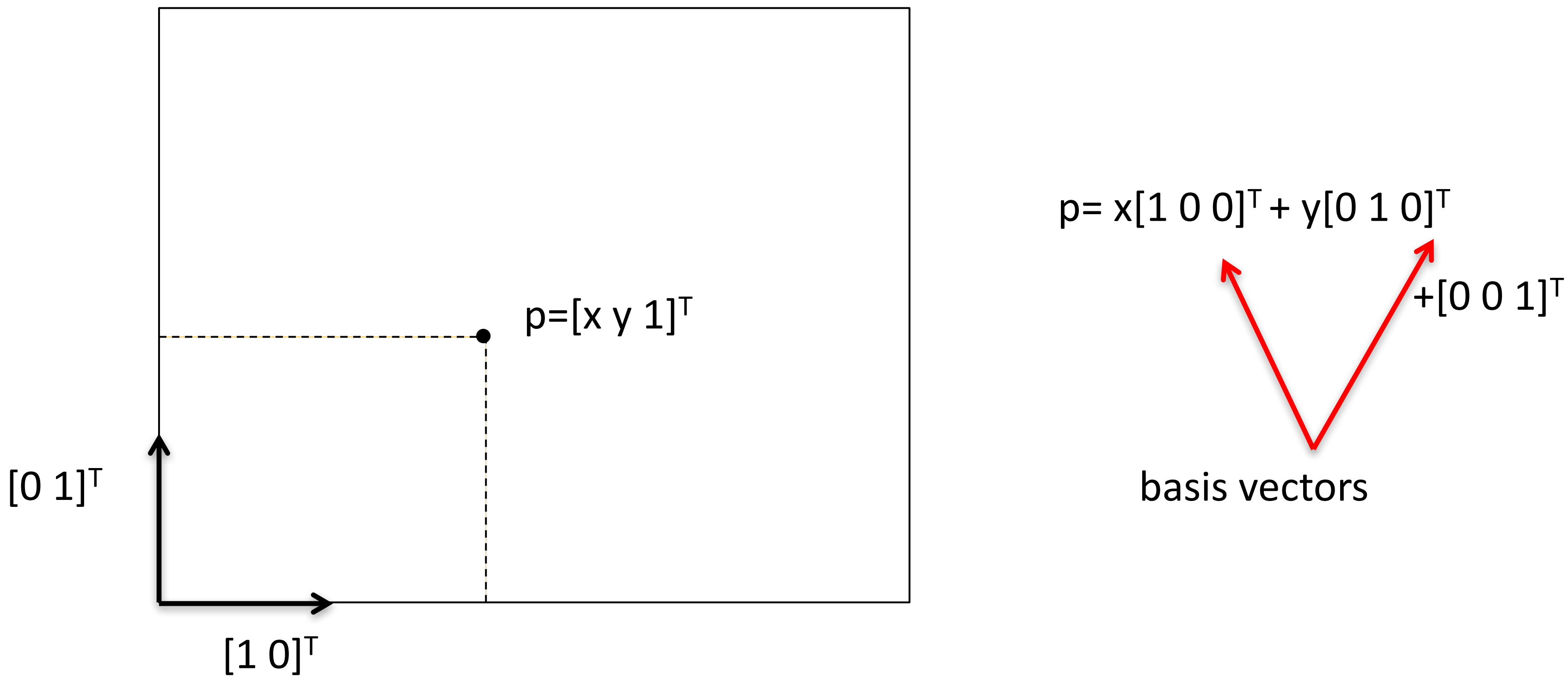
In 3D, shearing also "pushes things sideways" (in $X-$ and $Y-$direction) in a "fixed level" (the $Z-$value).

That "level" is a 2D subspace, i.e. a plane.

Ergo, we are doing a translation (in 2D) using matrix multiplication (in 3D).



Points as Homogeneous 2D Point Coords



Homogeneous coordinates in 2D: basic idea

We see: by adding a 3rd dimension to our 2D space, we can use matrix multiplication to create the following vectors:

$$M \begin{pmatrix} x \\ y \\ l \end{pmatrix} = \begin{pmatrix} x & + & x_t \\ y & + & y_t \\ l & & \end{pmatrix}$$

That's exactly what we want (for the first 2 coordinates). But:
How should the matrix M look like?
How about the two constants x_t, y_t?
And how are we dealing with this 3rd coordinate l?

Homogeneous coordinates in 2D: points

Translations in 2D can be represented as shearing in 3D by looking at the plane $z = 1$.

By representing all our 2D points (x, y) by 3D vectors $(x, y, 1)$, we can translate them about (x_t, y_t) using the following 3D shearing matrix:

$$\begin{pmatrix} 1 & 0 & x_t \\ 0 & 1 & y_t \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x + x_t \\ y + y_t \\ 1 \end{pmatrix}$$

Representing 2D transforms as a 3x3 matrix

Translate a point $[x\ y]^T$ by $[t_x\ t_y]^T$:

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Rotate a point $[x\ y]^T$ by an angle t :

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Scale a point $[x\ y]^T$ by a factor $[s_x\ s_y]^T$

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

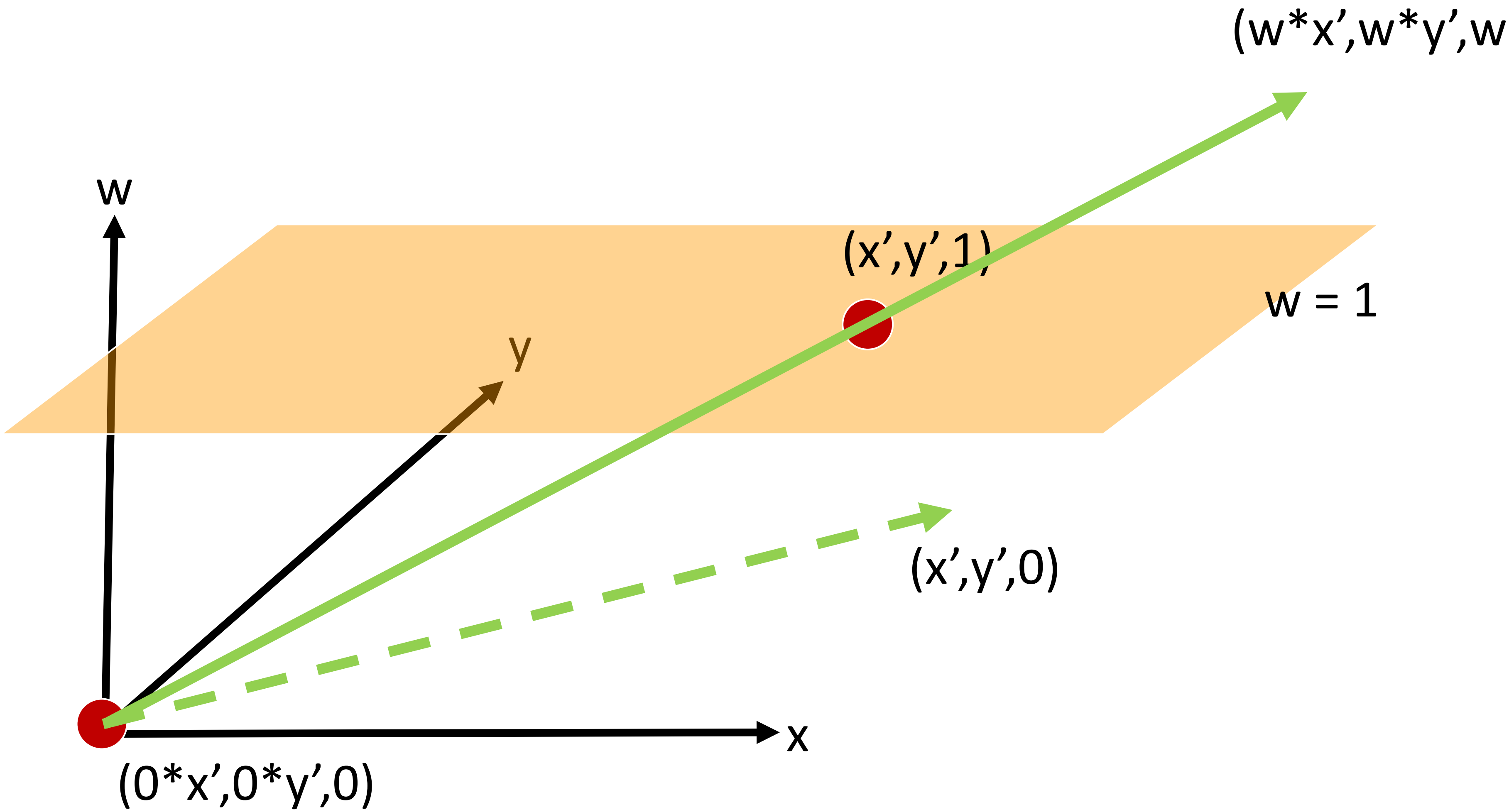
Cartesian ⇔ Homogeneous 2D Points

Cartesian $[x\ y]^T$ => Homogeneous $[x\ y\ 1]^T$

Homogeneous $[x\ y\ w]^T$ => Cartesian $[x/w\quad y/w\quad 1]^T$

Homogeneous points are equal if they represent the same Cartesian point. For eg. $[4\ {-6}\ 2]^T = [-6\ 9\ {-3}]^T$.

Geometric Intuition



Points at ∞ in Homogeneous Coordinates

$[x\ y\ w]^T$ with w=0 represent points at infinity, though with direction $[x\ y]^T$ and thus provide a natural representation for vectors, distinct from points in Homogeneous coordinates.

Line Equations in Homogeneous Coordinates

A line given by the equation

$$ax+by+c=0$$

can be represented in Homogeneous coordinates as:

l=[a b c] , making the line equation

$$l.p= [a\ b\ c][x\ y\ 1]^T = 0.$$

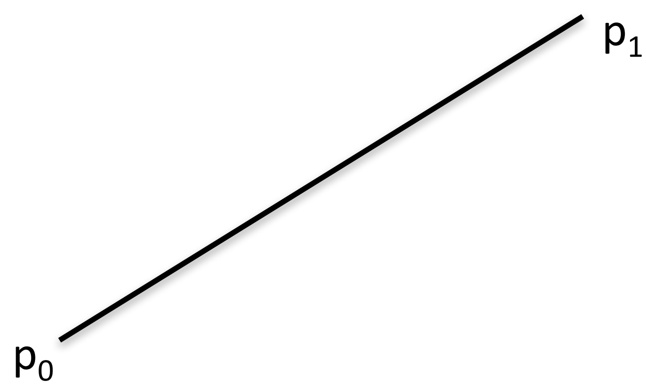
The Line Passing Through 2 Points

For a line l that passes through two points p_0, p_1

we have $l.p_0 = l.p_1 = 0$.

In other words we can write l using a cross product as:

$l = p_0 \times p_1$



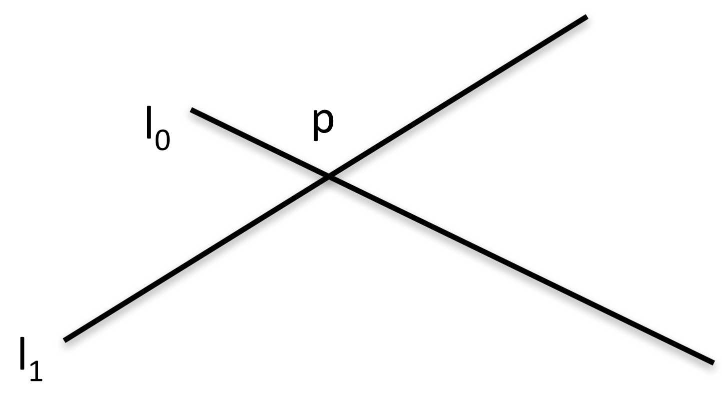
Point of intersection of 2 lines

For a point that is the intersection of two lines l_0, l_1

we have $p.l_0 = p.l_1 = 0$.

In other words we can write p using a cross product as:

$p = l_0 \times l_1$

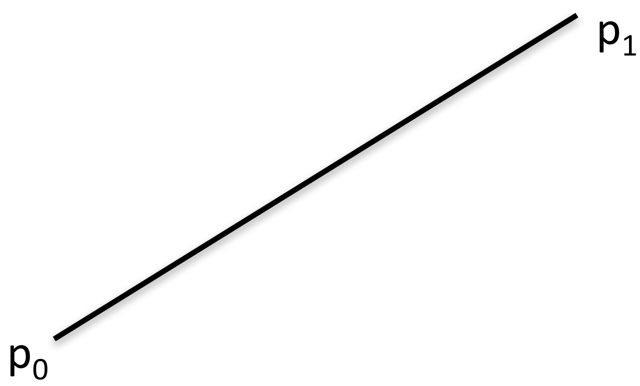


What happens when the lines are parallel?

A Line through 2 Points

For a line going through two points we have p_0, p_1

we have $p_0.l = p_1.l = 0$.



Properties of 2D transforms

…these 3x3 transforms have a variety of properties. most generally they map **lines** to **lines**. Such invertible **Linear** transforms are also called **Homographies**.

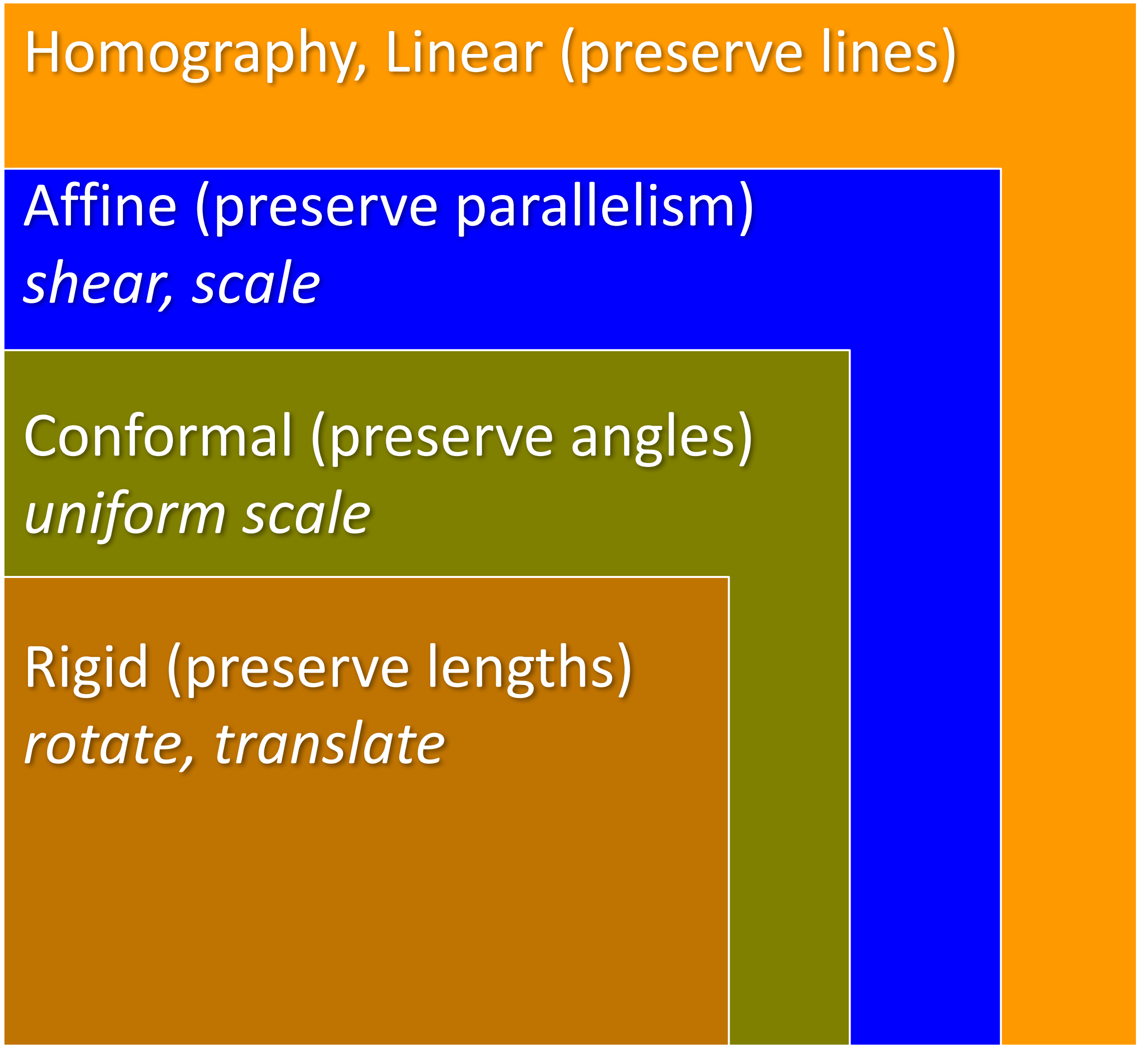
…a more restricted set of transformations also preserve parallelism in lines. These are called **Affine** transforms.

…transforms that further preserve the angle between lines are called **Conformal**.

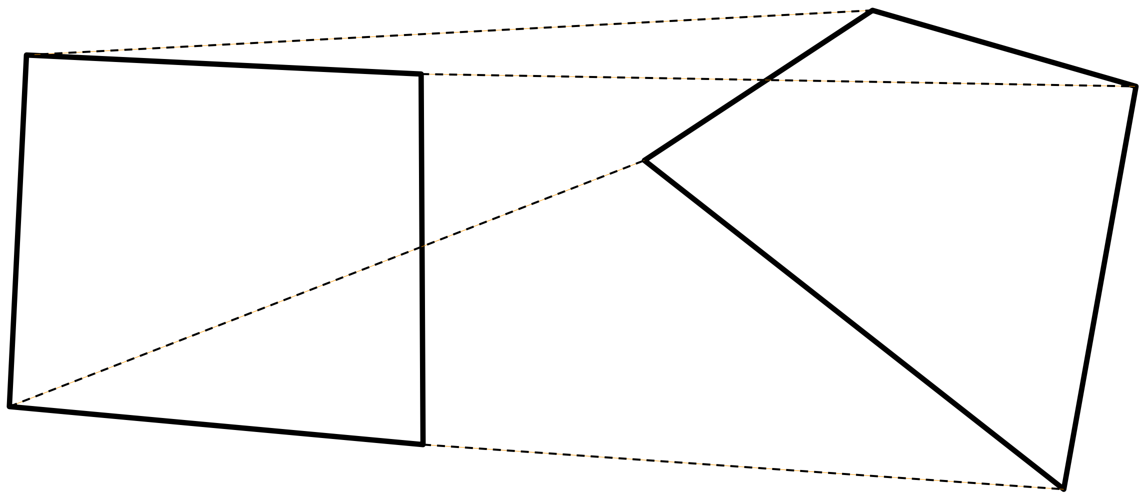
…transforms that additionally preserve the lengths of line segments are called **Rigid**.

Where do translate, rotate and scale fit into these?

Properties of 2D transforms



Homography: mapping four points



How does the mapping of 4 points uniquely define the 3x3 Homography matrix?

Homography: preserving lines

Show that if points p lie on some line l,
then their transformed points p’ also lie on some line l’.

Homography: preserving lines

Show that if points p lie on some line l,
then their transformed points p' also lie on some line l'.

Proof:
We are given that $l.p = 0$ and $p'=Hp$. Since H is invertible, $p=H^{-1}p'$.
Thus $l.(H^{-1}p')=0 \Rightarrow (lH^{-1}).p'=0$, or p' lies on a line $l'= lH^{-1}$.

QED

Affine: preserving parallel lines

What restriction does the Affine property impose on H?

If two lines are parallel their intersection point at infinity, is of the form $[x\ y\ 0]^T$.

If these lines map to lines that are still parallel, then $[x\ y\ 0]^T$ transformed must continue to map to a point at infinity or $[x'\ y'\ 0]^T$

i.e. $$[x'\ y'\ 0]^T = \begin{pmatrix} * & * & * \\ * & * & * \\ ? & ? & ? \end{pmatrix} [x\ y\ 0]^T$$

Affine: preserving parallel lines

What restriction does the Affine property impose on H?

If two lines are parallel their intersection point at infinity, is of the form $[x\ y\ 0]^T$.

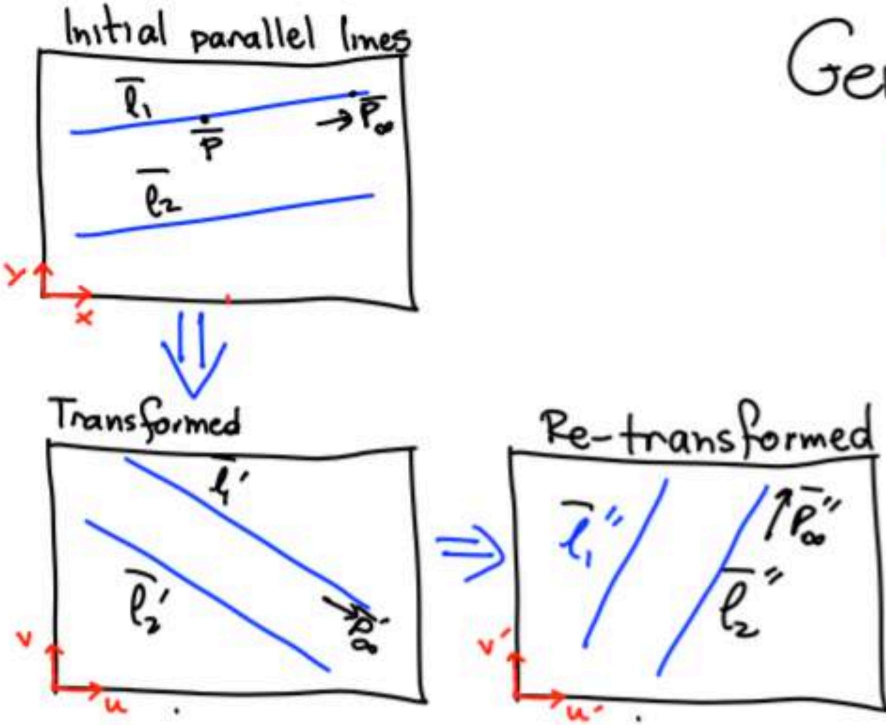
If these lines map to lines that are still parallel, then $[x\ y\ 0]^T$ transformed must continue to map to a point at infinity or $[x'\ y'\ 0]^T$

i.e. $$[x'\ y'\ 0]^T = \begin{pmatrix} A & t \\ 0\quad 0 & 1 \end{pmatrix} [x\ y\ 0]^T$$

In Cartesian co-ordinates Affine transforms can be written as:

$$p' = Ap + t$$

Affine Transformations: Composition



General form of matrix H

arbitrary 2x2 matrix — 2x1 vector

$$H = \left[\begin{array}{c|c} A & \bar{t} \\ \hline 0\ 0 & 1 \end{array}\right]$$

1. Affine transforms are closed under composition:

$$\left[\begin{array}{c|c} A_2 & \bar{t}_2 \\ \hline 0\ 0 & 1 \end{array}\right]\left[\begin{array}{c|c} A_1 & \bar{t}_1 \\ \hline 0\ 0 & 1 \end{array}\right] = \left[\begin{array}{c|c} A_2 A_1 + \bar{t}_2 [0\ 0] & A_2 \cdot \bar{t}_1 + \bar{t}_2 1 \\ \hline [0\ 0] A_1 + 1 \cdot [0\ 0] & [0\ 0] \bar{t}_1 + 1 \cdot 1 \end{array}\right] = \left[\begin{array}{c|c} A_2 A_1 & A_2 \bar{t}_1 + \bar{t}_2 \\ \hline 0\ 0 & 1 \end{array}\right]$$

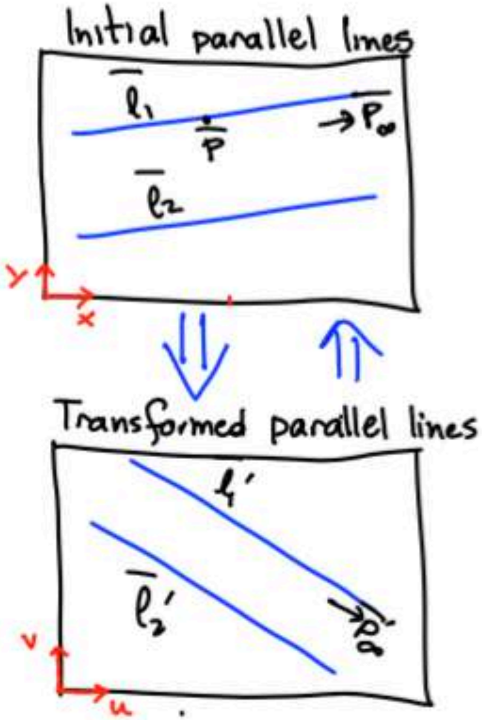
this is an affine transformation!

Affine properties: inverse

The inverse of an Affine transform is Affine.

- Prove it!

Affine Transformations: Inverse



General form of matrix H

arbitrary 2x2 matrix — 2x1 vector

$$H = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{A} & \bar{t} \\ \hline 0 & 0 & 1 \end{array}\right]$$

1. Affine transforms are closed under composition:

If H_1, H_2 are affine transform matrices, so is $H_1 \cdot H_2$.

2. The inverse H^{-1} of an affine transform H is affine

Proof: By definition, H^{-1} will map $\bar{p}_\infty'$ to $\bar{p}_\infty$. Since it preserves points at infinity, its matrix must have the above form. QED

Recall: Finding matrices

If we apply a linear transformation T to this vector, we get:

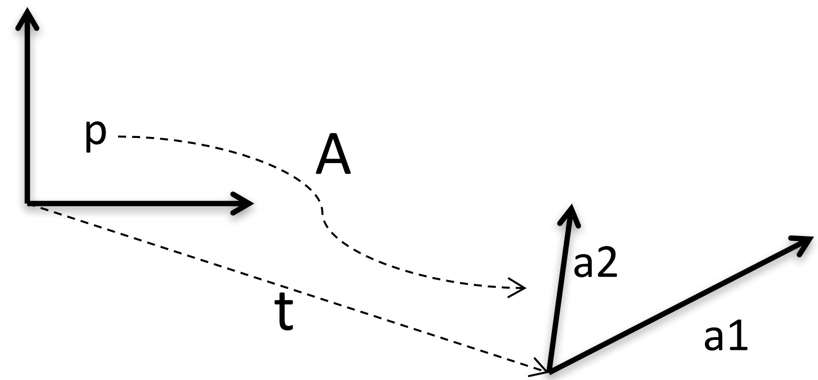
$$T(\begin{pmatrix} x \\ y \end{pmatrix}) = T(x\begin{pmatrix} 1 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \end{pmatrix}) = xT(\begin{pmatrix} 1 \\ 0 \end{pmatrix}) + yT(\begin{pmatrix} 0 \\ 1 \end{pmatrix})$$

$$= \begin{bmatrix} T\begin{pmatrix} 1 \\ 0 \end{pmatrix} & T\begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Transformation of a point is determined by a transformation of the basis vectors

Affine transform: geometric interpretation

A change of basis vectors and translation of the origin



point p in the local coordinates of a reference frame defined by <a1,a2,t> is

$$\begin{pmatrix} \begin{pmatrix} a1 & a2 \end{pmatrix} & \begin{bmatrix} t \end{bmatrix} \\ 0 \quad 0 & 1 \end{pmatrix}^{-1} \begin{bmatrix} p \end{bmatrix}$$

Composing Transformations

Any sequence of linear transforms can be collapsed into a single
3x3 matrix by concatenating the transforms in the sequence.

In general transforms DO NOT commute, however certain combinations of transformations are commutative…

try out various combinations of translate, rotate, scale.

Rotation about a fixed point

The typical rotation matrix, rotates points about the origin.
How do you rotate about specific point *q*

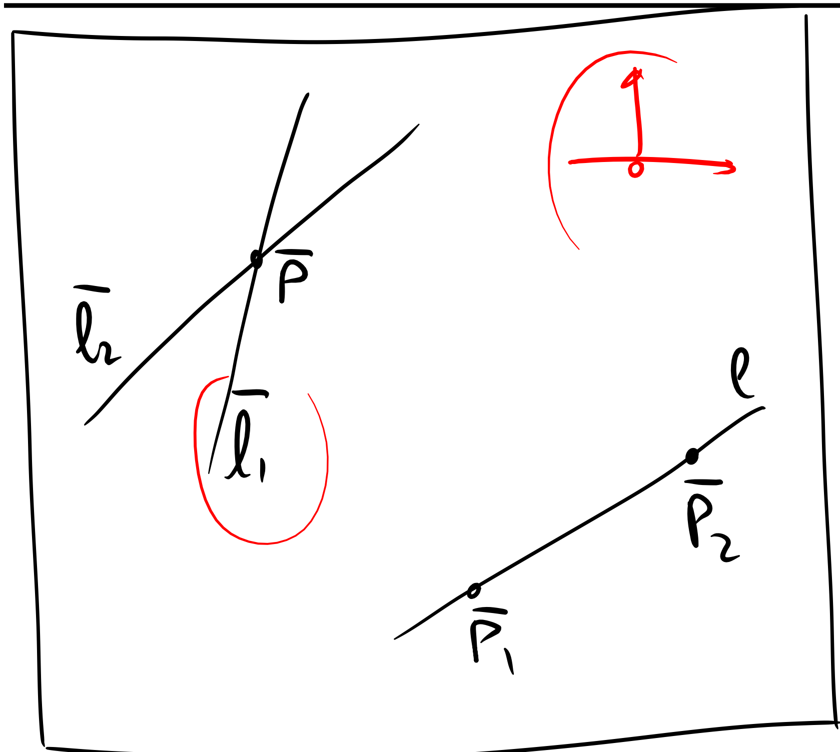
$$T_q \, R \, T_{-q}$$

Topic 4:

Coordinate-Free Geometry (CFG)

- A brief introduction & basic ideas

Doing Geometry Without Coordinates



- Style of expressing geometric objects & relations that avoids reliance on a coordinate system
- Useful in CG where we often deal with many coord systems

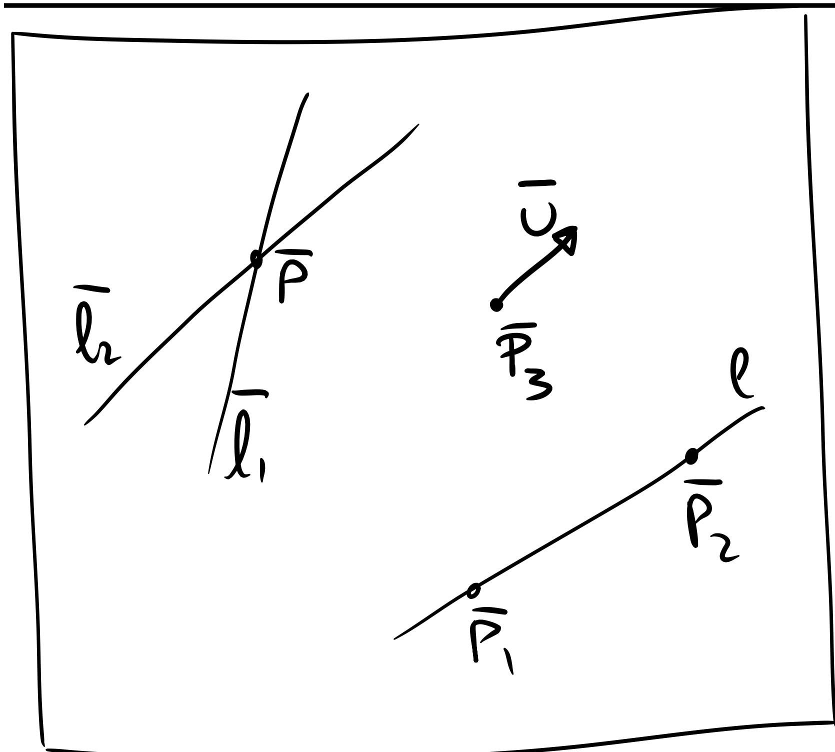
(#1) Intersection of 2 lines

$$\bar{p} = \bar{l}_1 \times \bar{l}_2$$

(#2) Line through 2 points

$$\bar{l} = \bar{p}_1 \times \bar{p}_2$$

CFG: Key Objects & their Homogeneous Repr.



Key objects:

- Points $\bar{p}$ $\quad \begin{bmatrix} u \\ v \\ w \end{bmatrix}$ $w \neq 0$
- Lines $\bar{l}$ $\quad \begin{bmatrix} a \\ b \\ c \end{bmatrix}$
- Vectors $\bar{u}$ $\quad \begin{bmatrix} \alpha \\ \beta \\ 0 \end{bmatrix}$

↑ homogenous coords

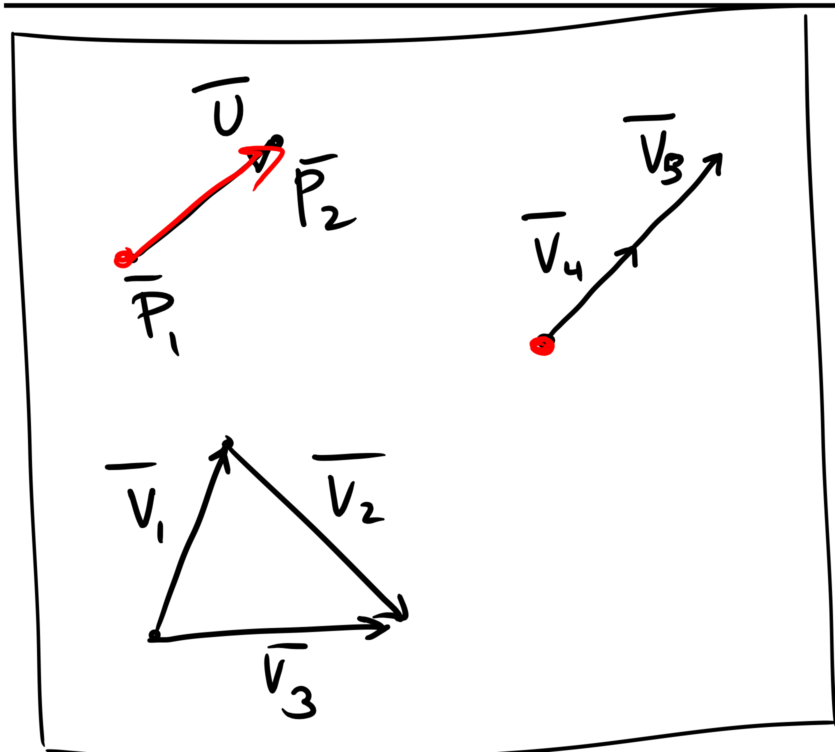
(#1) Intersection of 2 lines

$$\bar{p} = \bar{l}_1 \times \bar{l}_2$$

(#2) Line through 2 points

$$\bar{l} = \bar{p}_1 \times \bar{p}_2$$

CFG: Basic Geometric Operations



Key objects:

- Points $\bar{p}$ $\begin{bmatrix} x \\ y \\ w \end{bmatrix}, w \neq 0$
- Lines $\bar{\ell}$ $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$
- Vectors $\bar{u}$ $\begin{bmatrix} \alpha \\ \beta \\ 0 \end{bmatrix}$

(#1) Intersection of 2 lines

$$\bar{p} = \bar{\ell}_1 \times \bar{\ell}_2$$

(#2) Line through 2 points

$$\bar{\ell} = \bar{p}_1 \times \bar{p}_2$$

(#3) Point-vector addition

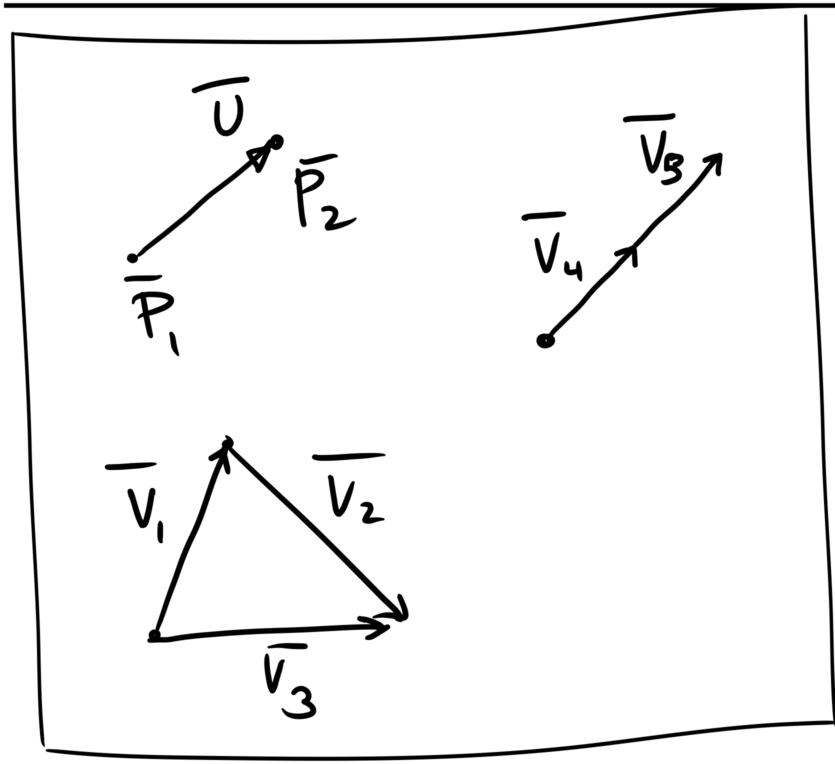
$$\bar{p}_1 + \bar{v} = \bar{p}_2$$

(#4) Vector-vector addition

$$\bar{v}_3 = \bar{v}_1 + \bar{v}_2$$

(#5) Vector scaling: $\bar{v}_5 = \lambda \bar{v}_4$

More CFG Ops: Linear Vector Combination



Key objects:

- Points $\bar{p}$ $\begin{bmatrix} x \\ y \\ w \end{bmatrix}$, $w \neq 0$
- Lines $\bar{\ell}$ $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$
- Vectors $\bar{u}$ $\begin{bmatrix} \alpha \\ \beta \\ 0 \end{bmatrix}$

(#3) Point-vector addition

$$\bar{p}_2 = \bar{p}_1 + \bar{v}$$

(#4) Vector-vector addition

$$\bar{v}_3 = \bar{v}_1 + \bar{v}_2$$

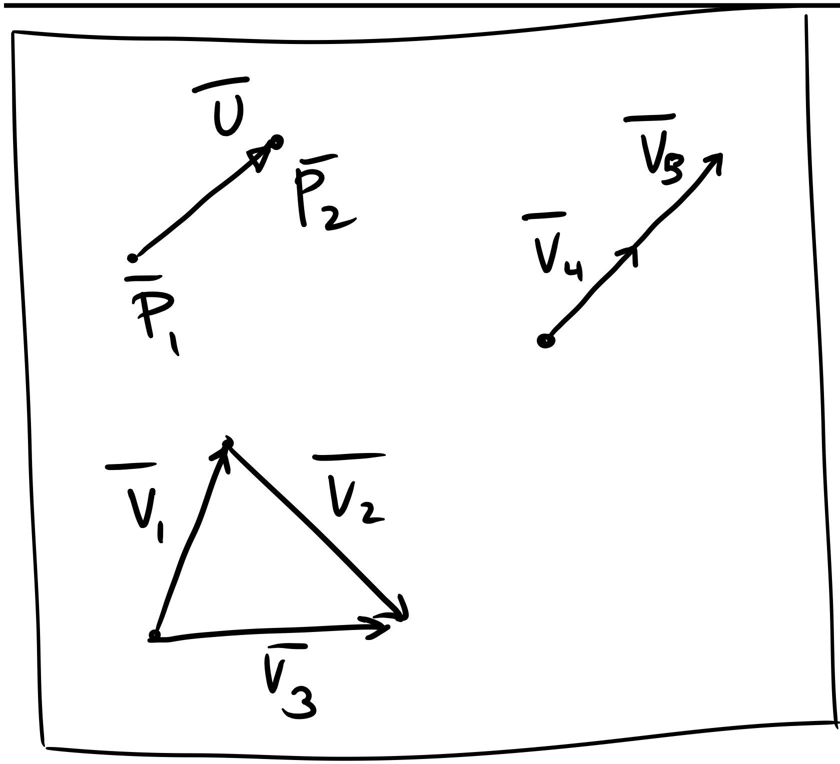
(#5) Vector scaling: $\bar{v}_5 = \lambda \bar{v}_4$

CAUTION: Addition possible only when 3rd homogeneous coordinate not affected

(#6) Linear vector combination

$$\bar{v} = \sum_{i=1}^{k} \lambda_k \bar{v}_i$$

More CFG Ops: Affine Point Combination



Key objects:

- Points $\bar{p}$ $\begin{bmatrix} x \\ y \\ w \end{bmatrix}, w \neq 0$
- Lines $\bar{\ell}$ $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$
- Vectors $\bar{v}$ $\begin{bmatrix} \alpha \\ \beta \\ 0 \end{bmatrix}$

(#3) Point-vector addition

$$\bar{p}_2 = \bar{p}_1 + \bar{v}$$

(#7) Point subtraction

$$\bar{v} = \bar{p}_1 - \bar{p}_2$$ (ONLY if last coord is same)

(#8) Affine point combination

$$\bar{p} = \sum_{i=1}^{k} \lambda_i \bar{p}_i$$

only when all $\bar{p}_i$ have same 3rd coord AND

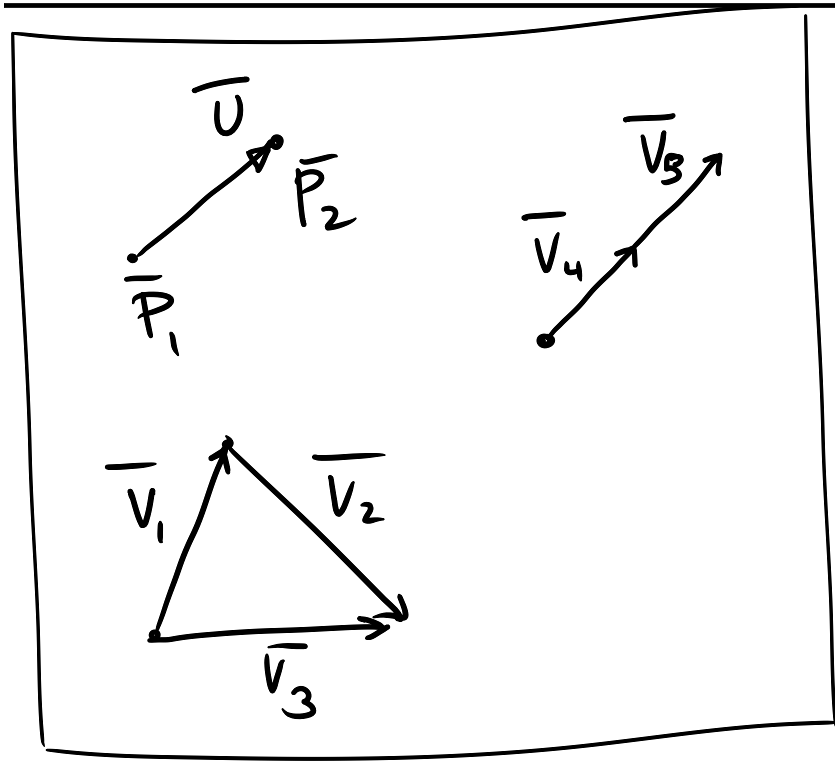
$$= \bar{p}_1 + (\lambda_1 - 1)\bar{p}_1 + \sum_{i=2}^{k} \lambda_i \bar{p}_i$$

$\sum_{i=1}^{k} \lambda_i = 1$ — i.e. circled expression is a vector ⇒ reduces to (#3)

OR

$\sum_{i=1}^{k} \lambda_i = 0$ — i.e. circled expression is a point $\bar{q}$ with same 3rd coord ⇒ reduces to (#7)

More CFG Ops: Operations w/ Scalar Result



Key objects:

- Points $\bar{p}$ $\begin{bmatrix} x \\ y \\ w \end{bmatrix}, w \neq 0$
- Lines $\bar{\ell}$ $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$
- Vectors $\bar{v}$ $\begin{bmatrix} \alpha \\ \beta \\ 0 \end{bmatrix}$

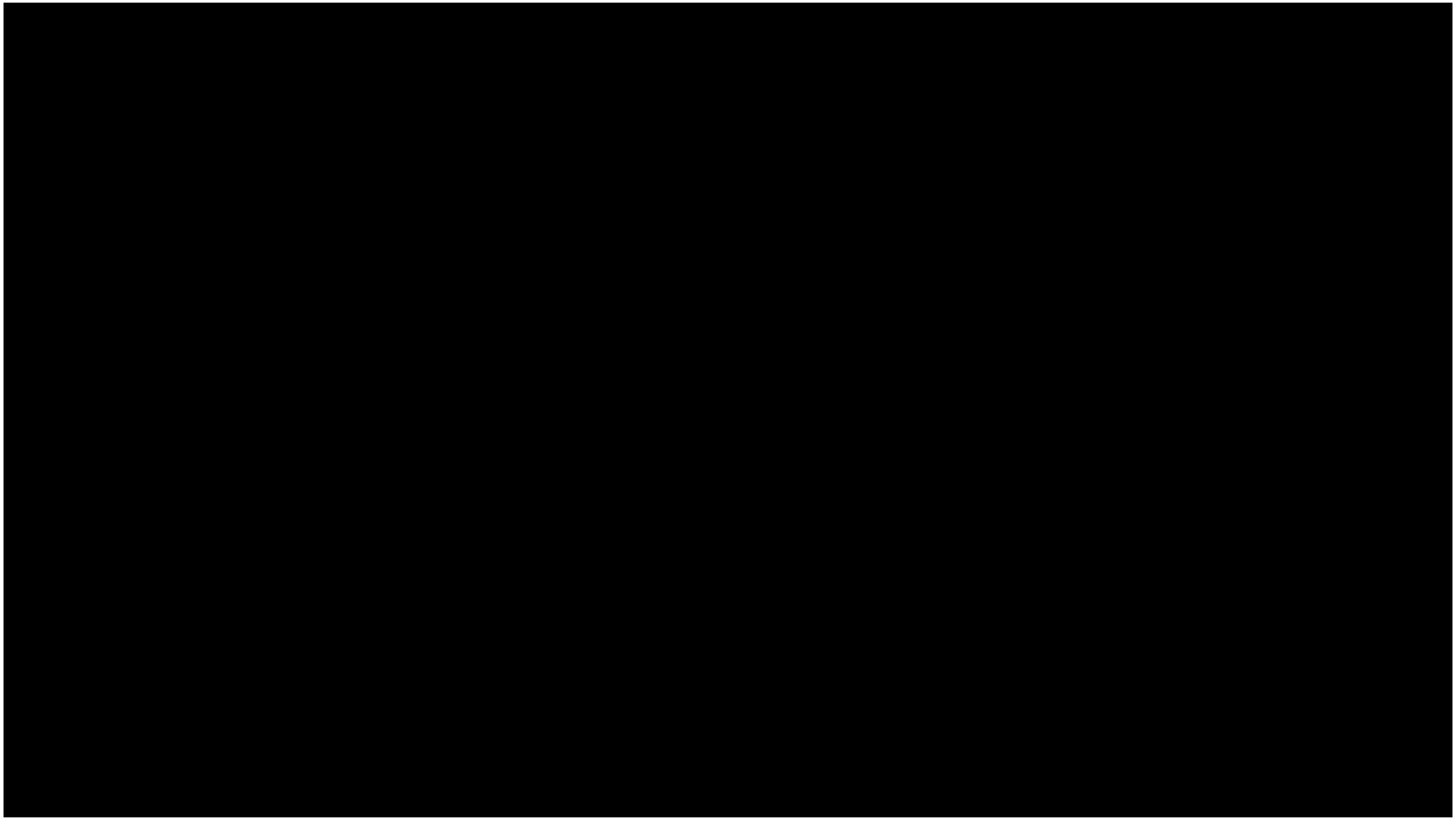
(#9) $\|\bar{v}\|$ magnitude of vector $\bar{v}$

(#10) $\bar{v}_1 \cdot \bar{v}_2$ dot product of two vectors (also written as $(\bar{v}_1)^T (v_2)$ in matrix notation)

(#11) $\bar{\ell} \cdot \bar{p}$ dot product of a line and a point

Lecture 3 Starts Here

Showtime



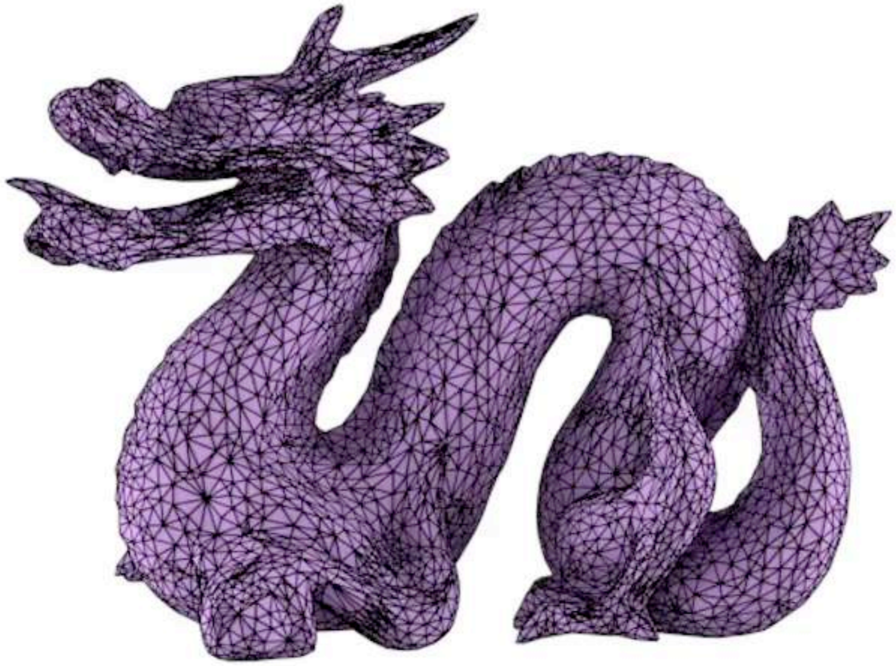
SIGGRAPH Submissions

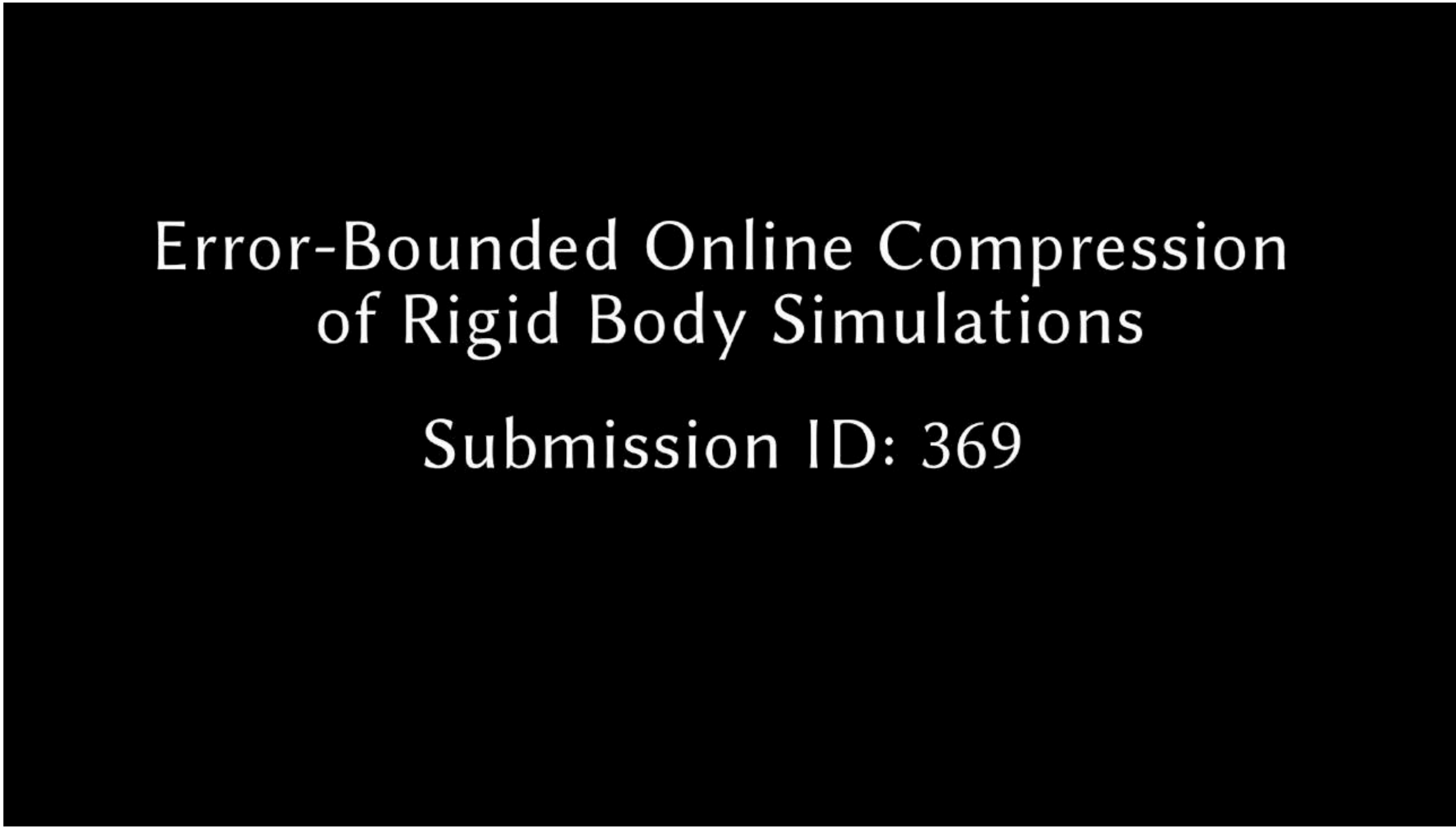
Designing Volumetric Truss Structures for Computational Fabrication

Submission #0131

Autodef: Nonlinear Subspace Simulation for Large Deformation Elastodynamics

ID 0363




Error-Bounded Online Compression
of Rigid Body Simulations
Submission ID: 369

https://s2018.siggraph.org/conference/conference-overview/student-volunteers/

Questions about the Midterm

If you have a valid, **documented** reason for missing the midterm exam, your final exam will be worth 50%

Midterm will be in tutorials so if you are in my tutorial that means **Monday, February 12**

Questions about the Assignment

Please contact the TAs via email at csc418tas@cs.toronto.edu

Assignment 2 is not due during reading week. It will be due the **Monday after reading week February 26th.**

Topic 5:

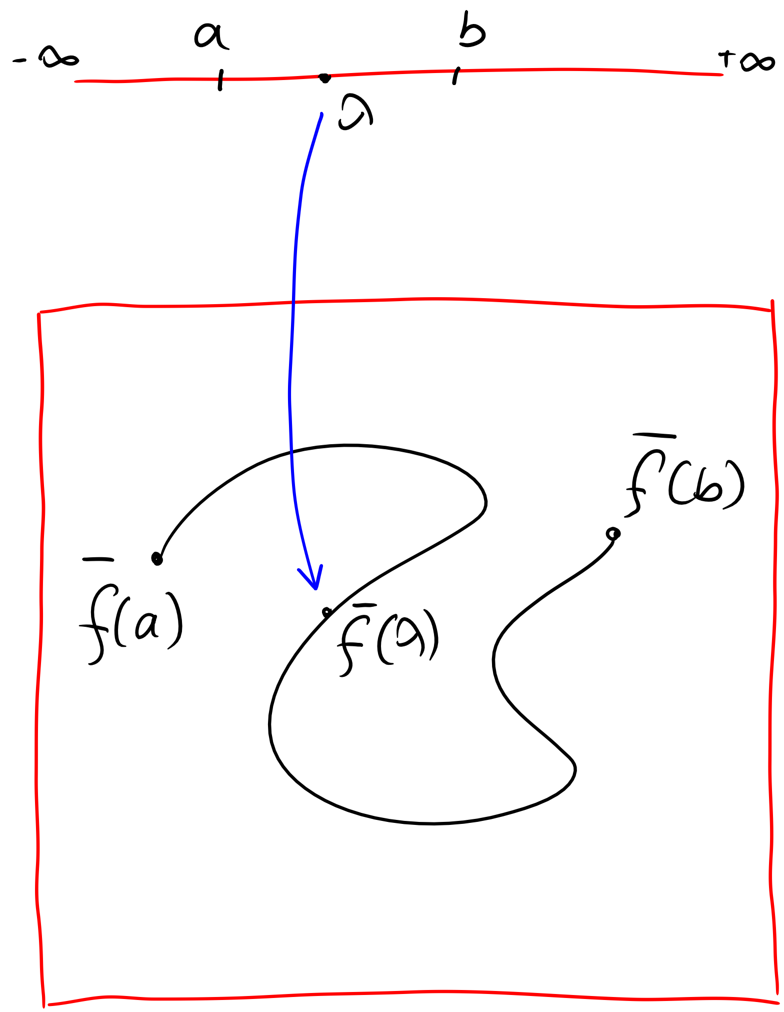
3D Objects

- General curves & surfaces in 3D
- Normal vectors, surface curves & tangent planes
- Implicit surface representations
- Example surfaces:
  surfaces of revolution, bilinear patches, quadrics

Reminder: Curves in 2D

Space of the curve parameter (1D)

$\lambda \in (a,b) \subseteq \mathbb{R}$



$\bar{f}(\lambda) \in \mathbb{R}^2$

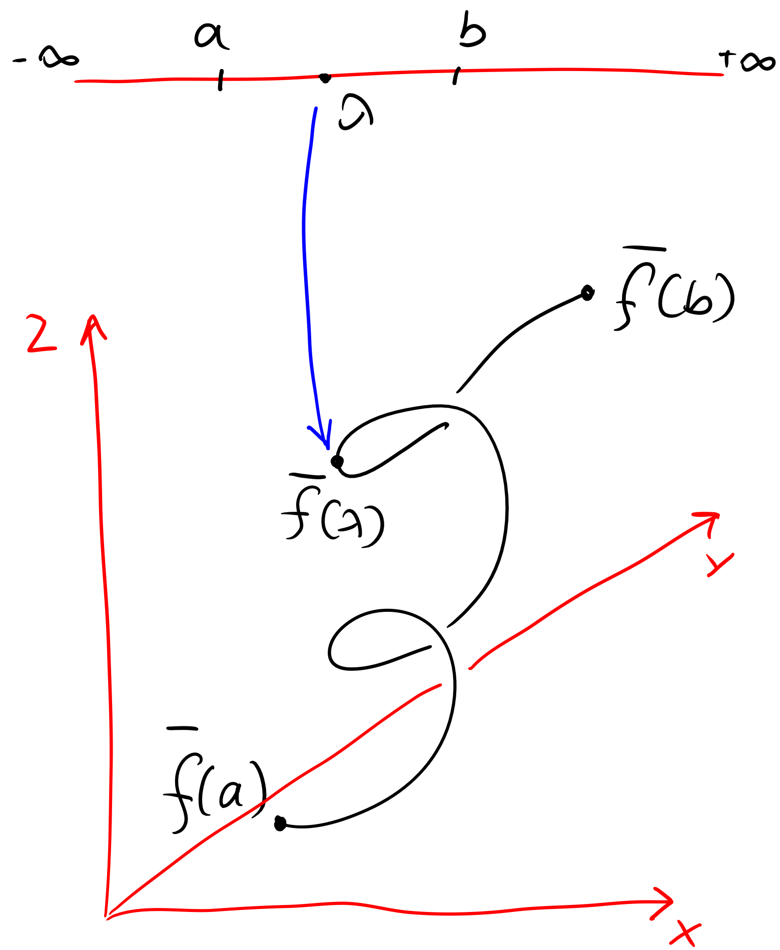
where

$\bar{f}(\lambda) = \big(x(\lambda), y(\lambda)\big)$

Space of the curve (2D)

Curves in 3D

Space of the curve parameter (1D)



$\lambda \in (a,b) \subseteq \mathbb{R}$

$\bar{f}(\lambda) \in \mathbb{R}^3$

where

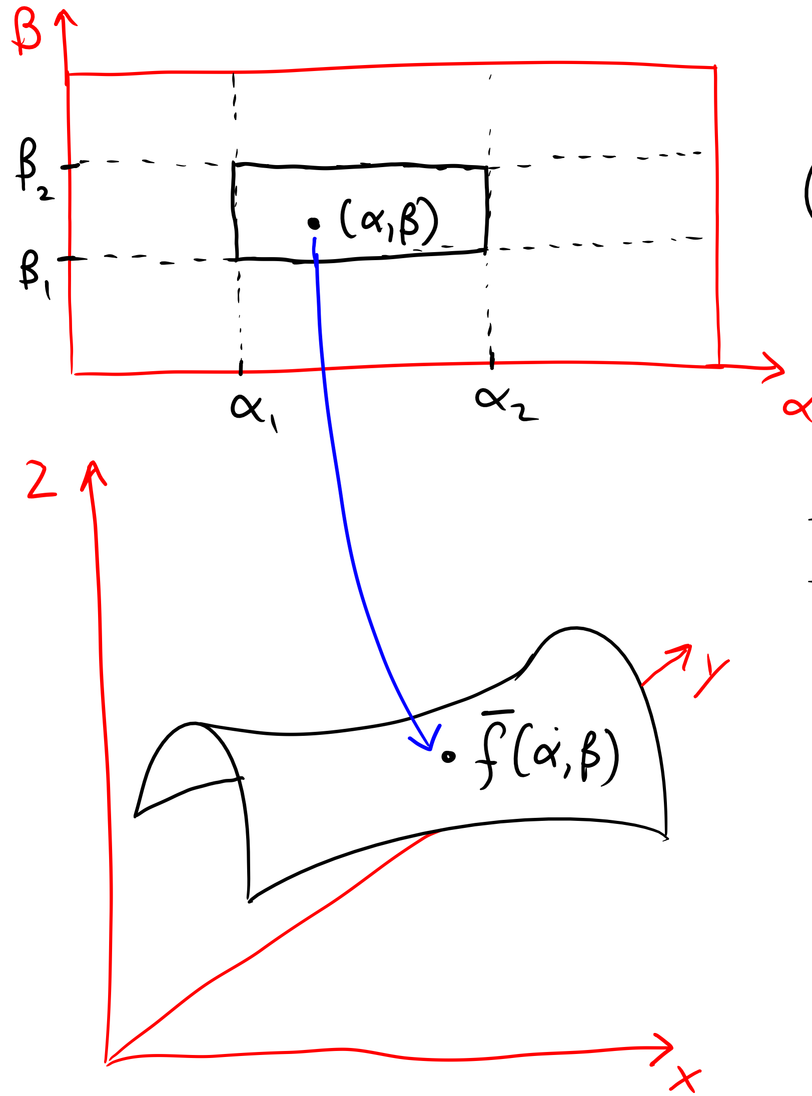
$\bar{f}(\lambda) =$

$\left(x(\lambda), y(\lambda), z(\lambda) \right)$

Space of the curve (3D)

Surfaces in 3D

Space of the surface parameters



Space of the surface (3D)

$(\alpha,\beta) \in [\alpha_1,\alpha_2] \times [\beta_1,\beta_2]$

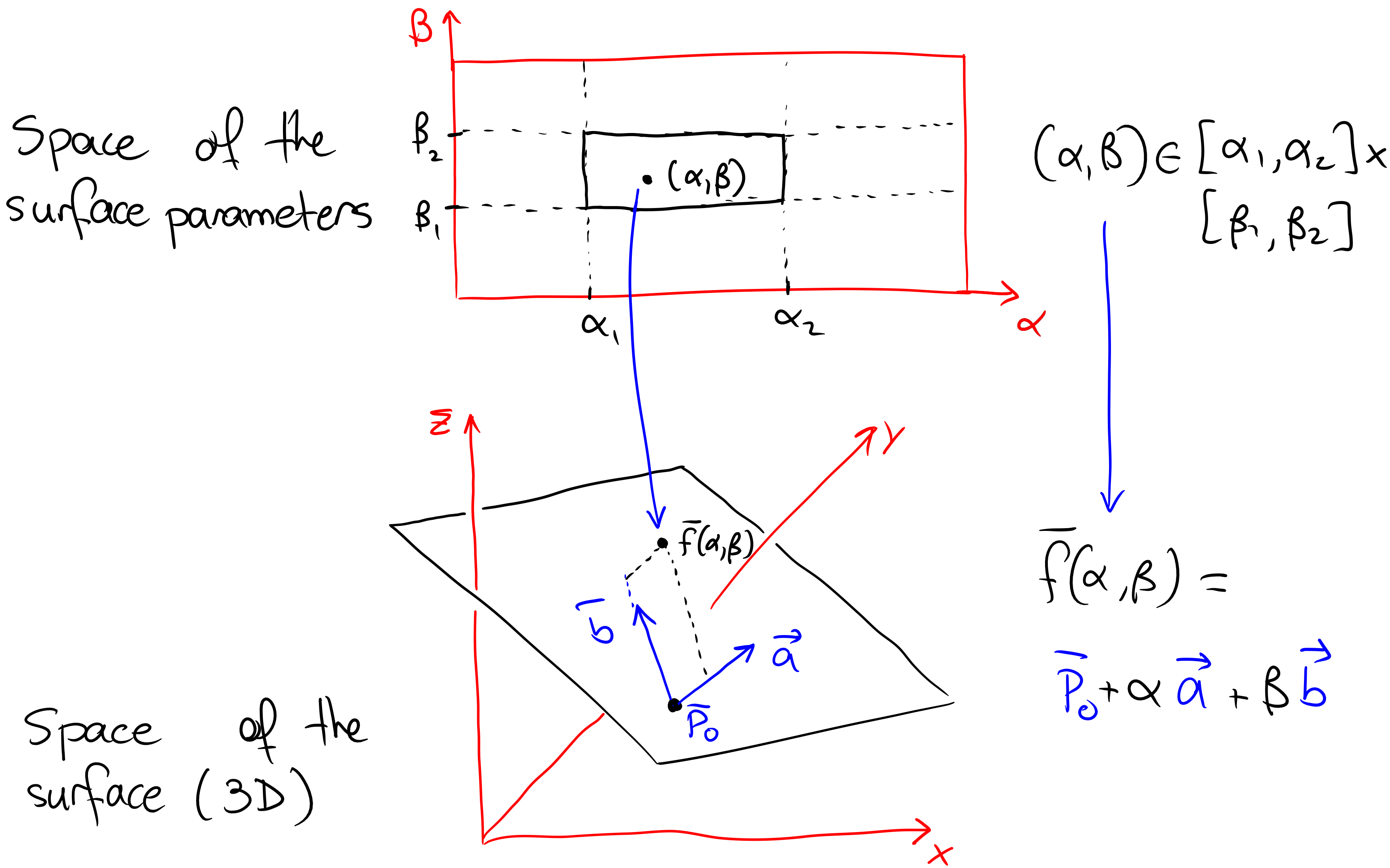
$\downarrow$

$\bar{f}(\alpha,\beta) \in \mathbb{R}^3$

where

$$\bar{f}(\alpha,\beta) = \big(x(\alpha,\beta),\ y(\alpha,\beta),\ z(\alpha,\beta)\big)$$

Surface Example: Planes in 3D



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Tangent / Normal vectors of 2D curves

Explicit: y=f(x). Tangent is dy/dx.

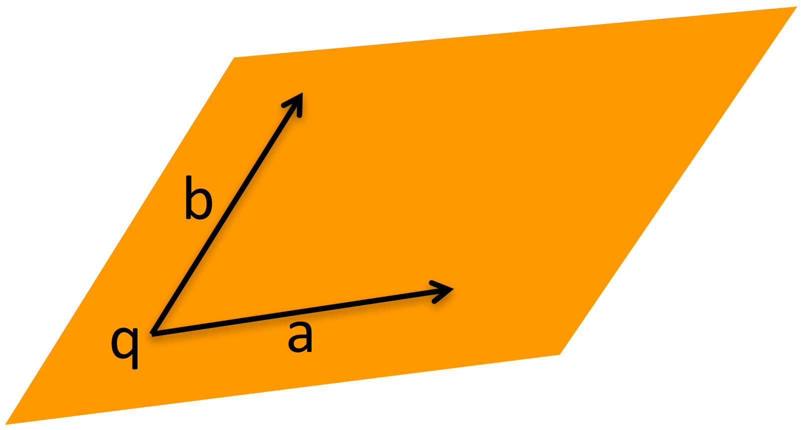
Parametric: $x=f_x(t)$, $y=f_y(t)$. Tangent is (dx/dt, dy/dt)

Implicit: f(x,y) = 0. Normal is gradient(f).
direction of max. change

Given a tangent or normal vector in 2D how do we compute the other?

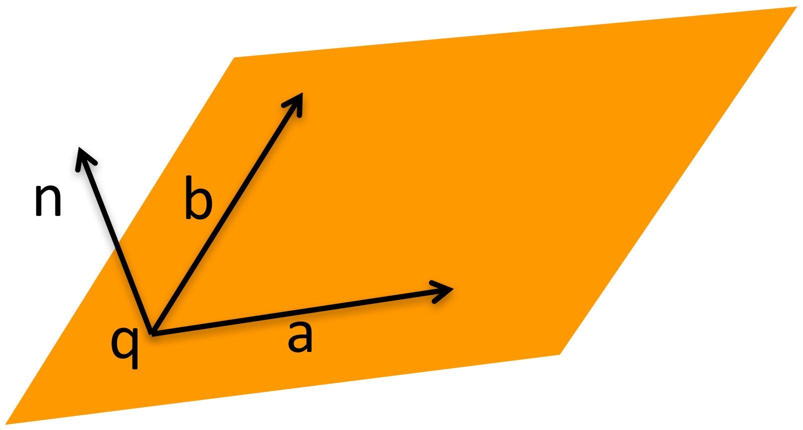
What about in 3D?

Normal vector of a plane



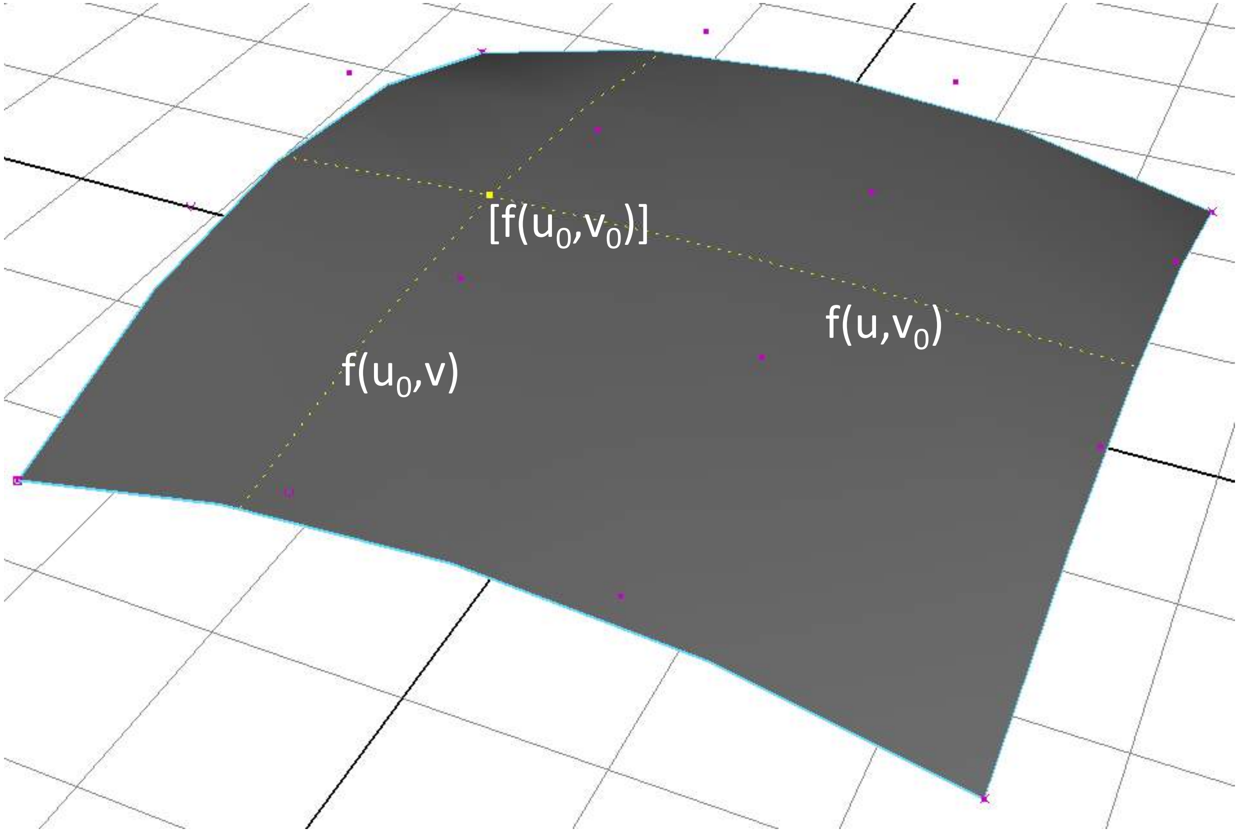
p(s,t)= q + as +tb

Normal vector of a plane

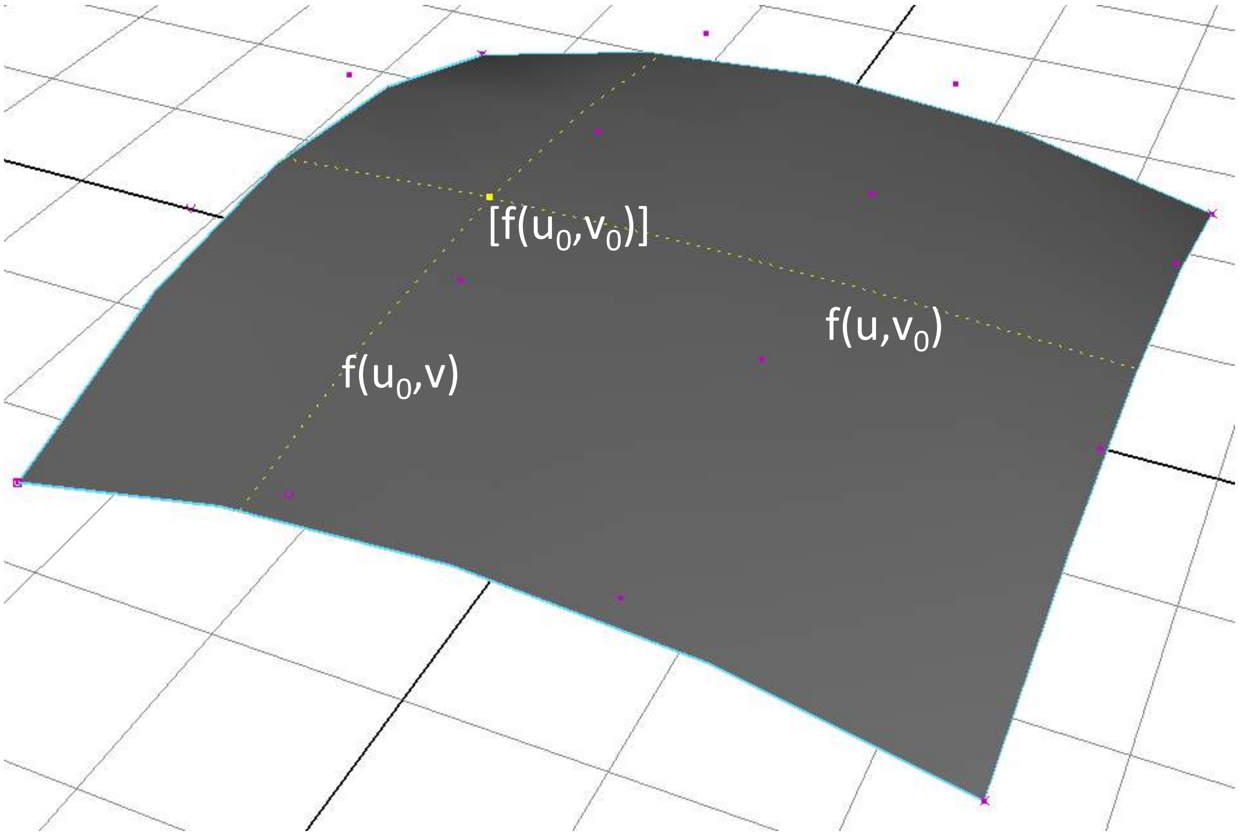


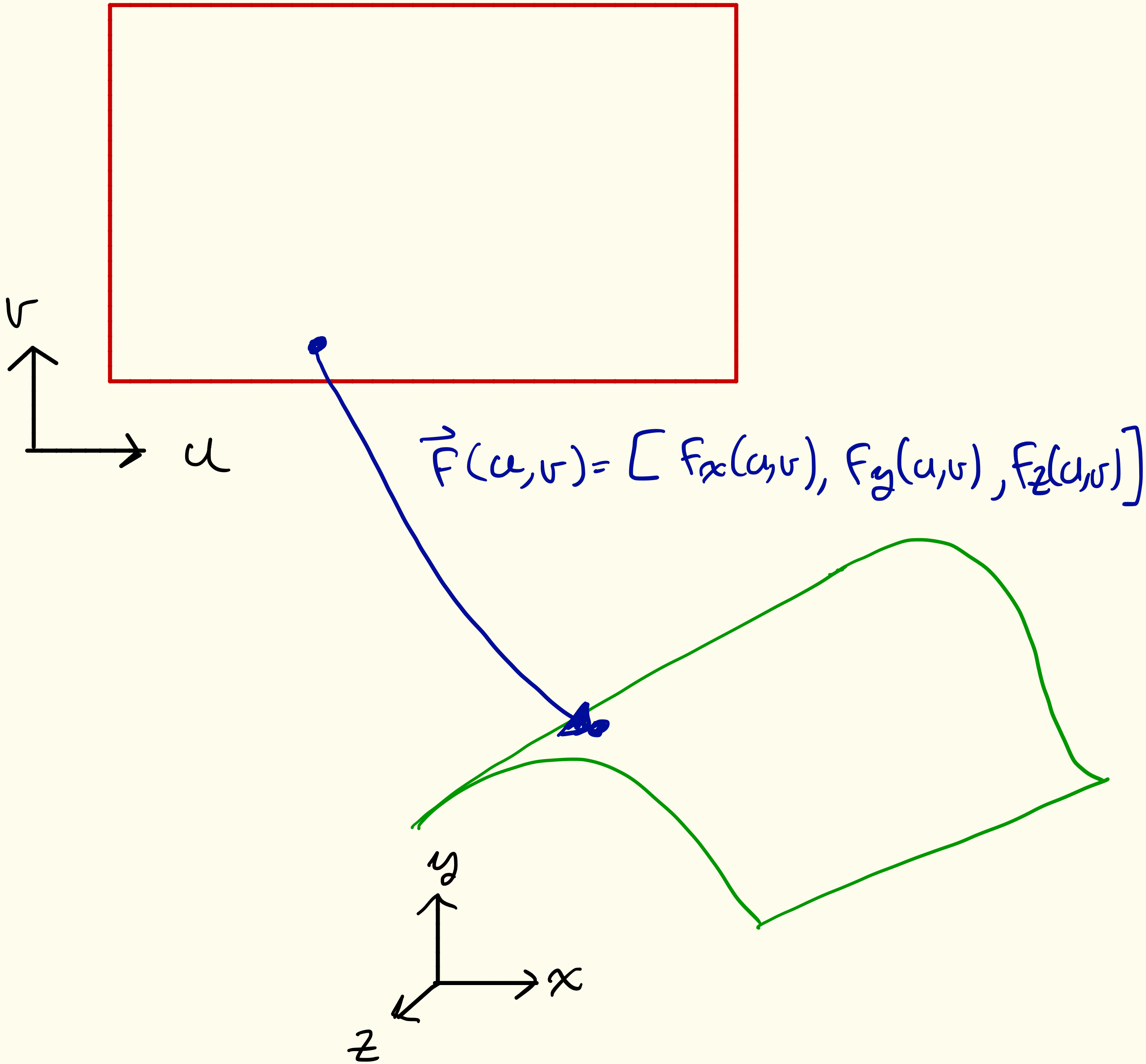
n=aXb

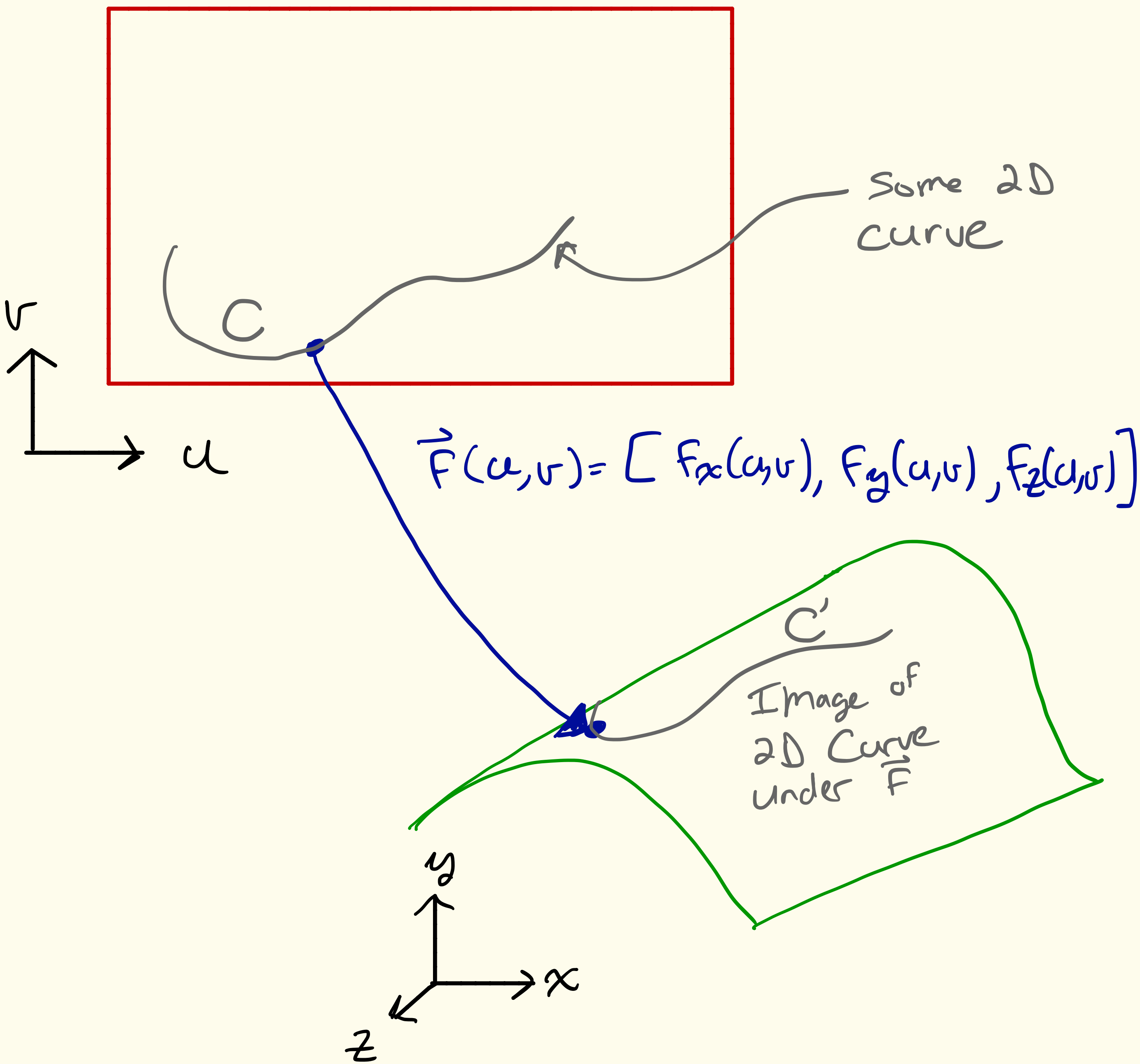
Normal vector of a parametric surface



Tangent vectors of a parametric surface




v
u
$\vec{F}(u,v) = [F_x(u,v), F_y(u,v), F_z(u,v)]$
y
x
z


Some 2D curve
C
v
u
F(u,v) = [Fx(u,v), Fy(u,v), Fz(u,v)]
C'
Image of 2D Curve under F
y
x
z

Now some more math

$$C(s) = [u(s), v(s)]$$

$$C'(s) = \vec{F}(u(s), v(s))$$

$$C'(s+\Delta s) = C'(t) + \frac{\partial \vec{F}}{\partial u}\frac{\partial u}{\partial s}\Delta s + \frac{\partial \vec{F}}{\partial v}\frac{\partial v}{\partial s}\Delta s$$

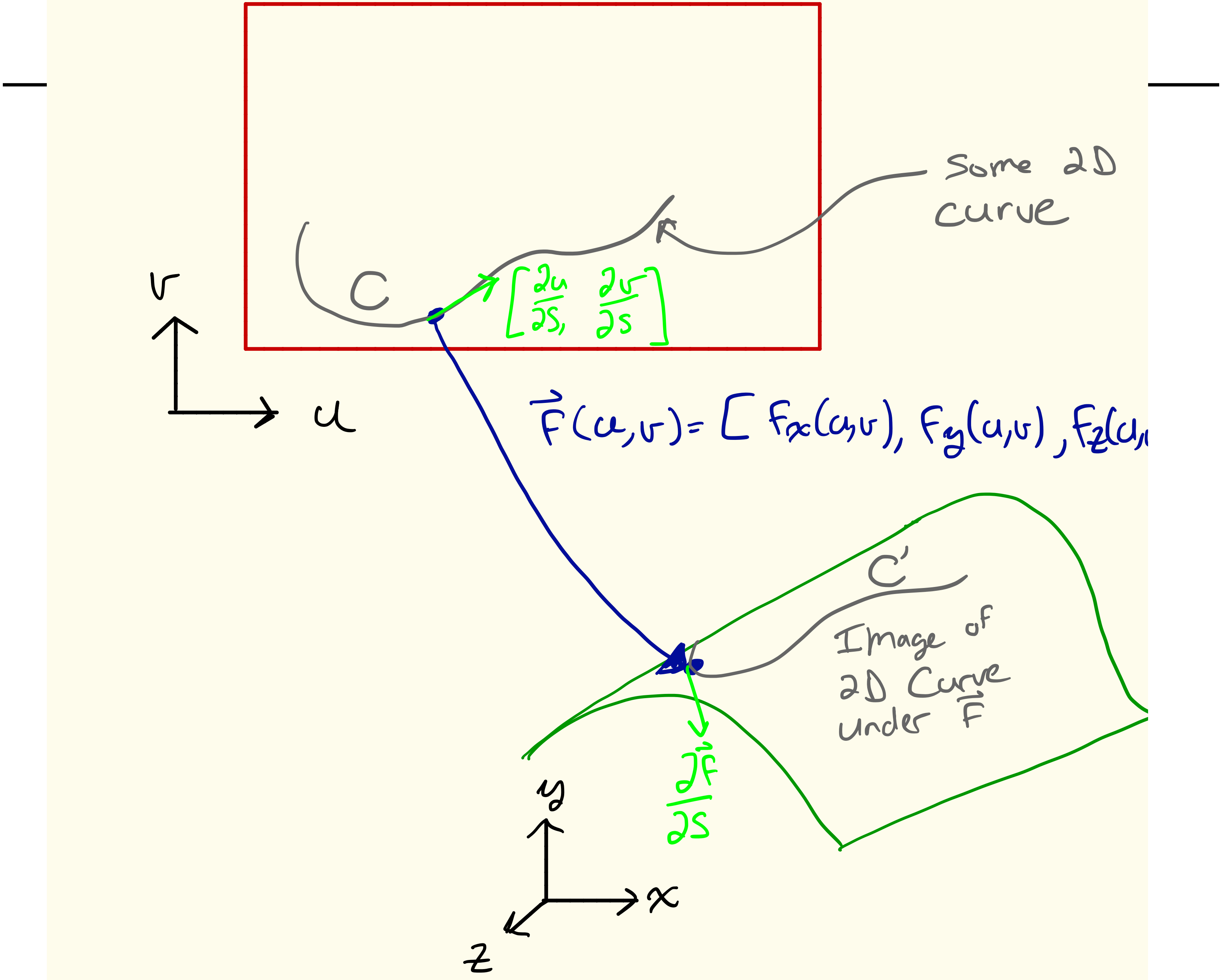
$$\frac{C'(s+\Delta s) - C'(t)}{\Delta s} = \begin{bmatrix} \frac{\partial \vec{F}}{\partial u} & \frac{\partial \vec{F}}{\partial v} \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial s} \\ \frac{\partial v}{\partial s} \end{bmatrix}$$

vector tangential to surface

Gradient of 3D Parametric Surface

tangent of C

Independent of C!



Some 2D curve
C
[∂u/∂s, ∂v/∂s]
v
u
F(u,v) = [Fx(u,v), Fy(u,v), Fz(u,
C'
Image of 2D Curve under F
∂F/∂s
y
x
z

$$C(s) = [u(s), v(s)]$$

$$C'(s) = \vec{F}(u(s), v(s))$$

$$C'(s+\Delta s) = C'(t) + \frac{\partial \vec{F}}{\partial u}\frac{\partial u}{\partial s}\Delta s + \frac{\partial \vec{F}}{\partial v}\frac{\partial v}{\partial s}\Delta s$$

$$\underbrace{\frac{C'(s+\Delta s) - C'(t)}{\Delta s}}_{\text{vector tangential to surface}} = \underbrace{\begin{bmatrix} \frac{\partial \vec{F}}{\partial u} & \frac{\partial \vec{F}}{\partial v} \end{bmatrix}}_{\text{Gradient of 3D Parametric Surface}} \underbrace{\begin{bmatrix} \frac{\partial u}{\partial s} \\ \frac{\partial v}{\partial s} \end{bmatrix}}_{\text{tangent of } C}$$

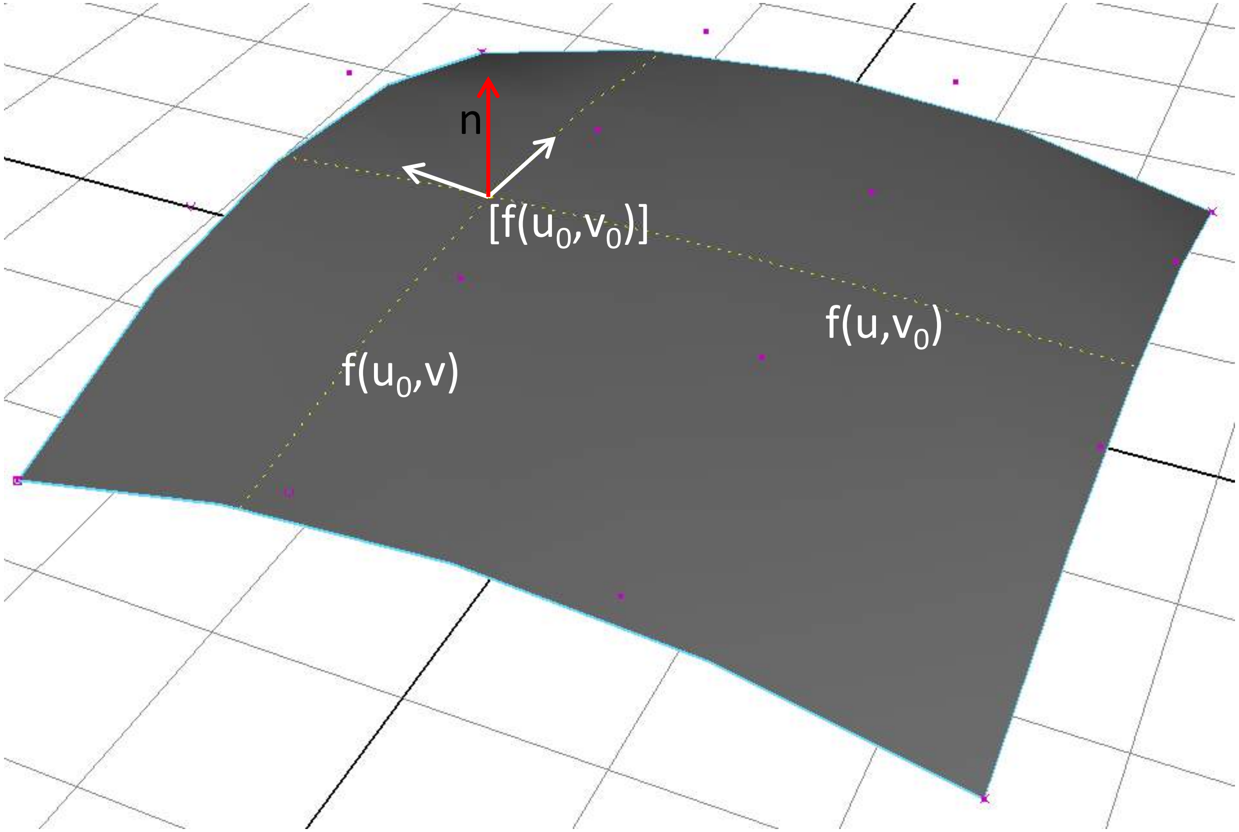
vector tangential to surface

Gradient of 3D Parametric Surface

tangent of C

Independent of C !

Normal vector of a parametric surface



$$n = f'(u_0, v) \times f'(u, v_0)$$

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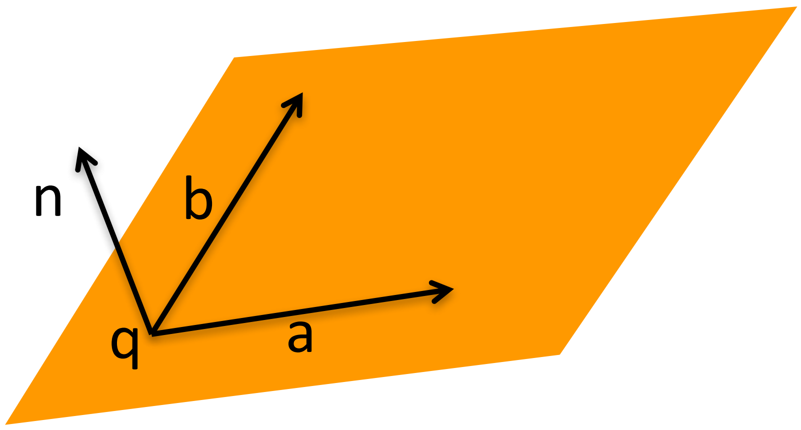
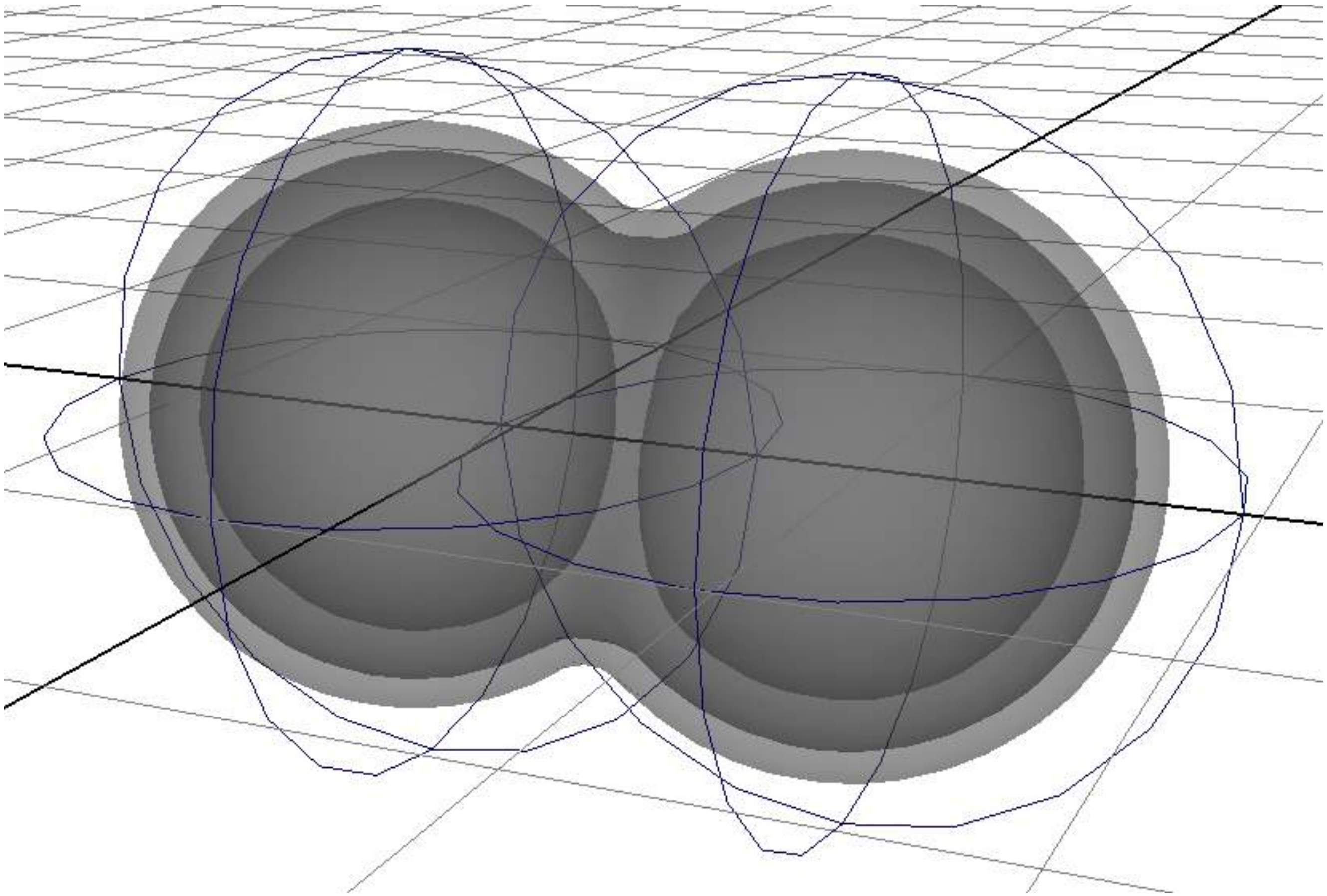
Implicit function of a plane



$$f(p) = (p-q).n=0$$

Implicit function: level sets

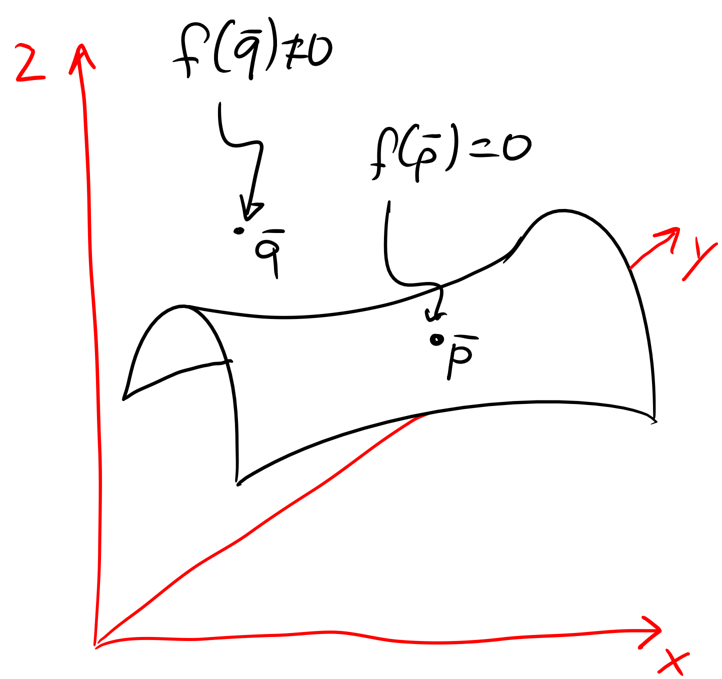


Representing Surfaces by an Implicit Function

- Representation consists of a scalar function $f: \mathbb{R}^3 \rightarrow \mathbb{R}$ (called the Implicit Function)
- Surface defined as the set

$$\alpha_0 = \{\bar{p} \in \mathbb{R}^3 \mid f(\bar{p}) = 0\}$$

- Intuitively, f can be thought of as measuring a "distance" to the surface
- Typically, f does NOT measure Euclidean distance to the surface



Example: The Implicit Function of a Plane

- Representation consists of a scalar function $f: \mathbb{R}^3 \rightarrow \mathbb{R}$
- Surface defined as the set

$$\alpha_0 = \{ \bar{p} \in \mathbb{R}^3 \mid f(\bar{p}) = 0 \}$$

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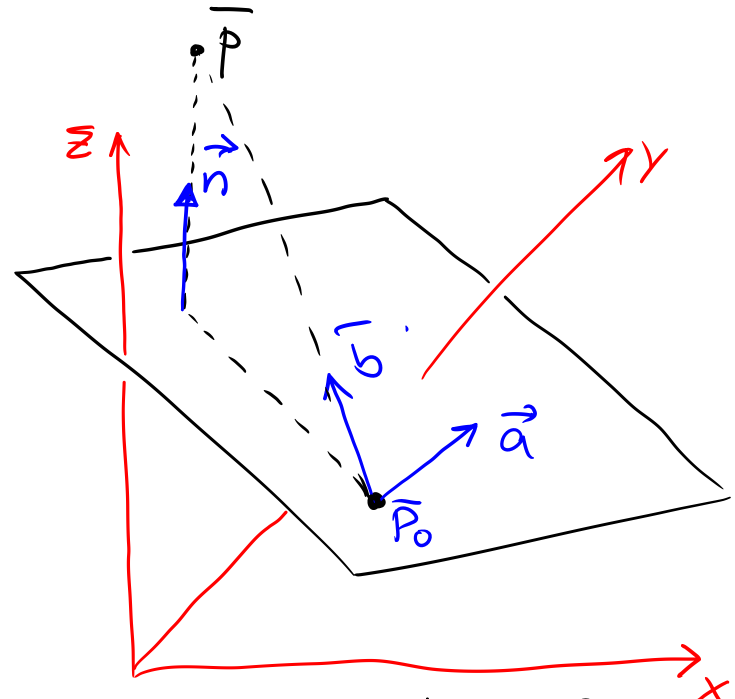


Example: Implicit function for a plane through $\bar{p}_0$ with normal $\vec{n}$:

$$f(\bar{p}) = (\bar{p} - \bar{p}_0) \cdot \vec{n}$$

Surface Normals from the Implicit Function

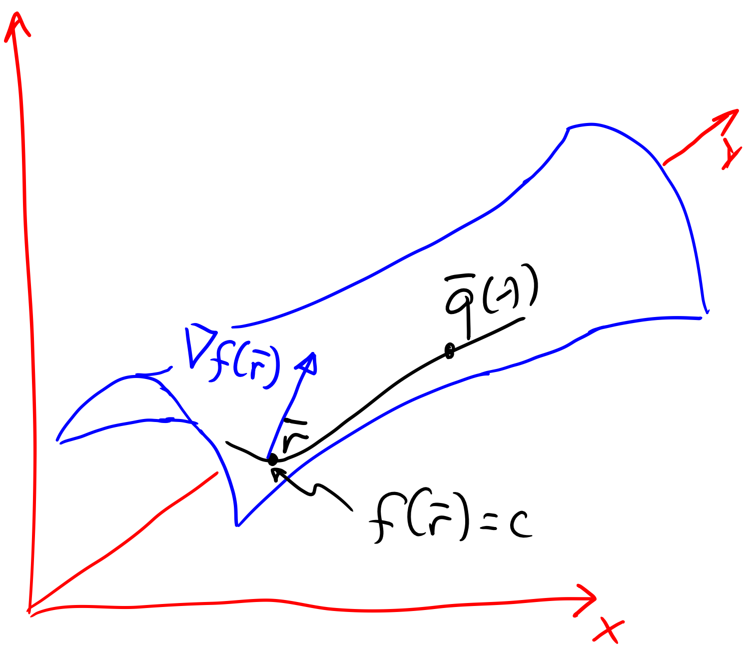
Proof: Let $\bar{q}(\lambda) = (x(\lambda), y(\lambda), z(\lambda))$ be a curve on the surface α_c with $\bar{q}(0) = \bar{r}$.

$$\Leftrightarrow \quad \forall \lambda \quad f(\bar{q}(\lambda)) = c$$

$$\Leftrightarrow \quad \frac{df}{d\lambda}(\bar{q}(0)) = 0$$

$$\Leftrightarrow \quad \frac{\partial f}{\partial x}\frac{dx}{d\lambda} + \frac{\partial f}{\partial y}\cdot\frac{dy}{d\lambda} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial \lambda} = 0$$

$$\Leftrightarrow \quad \nabla f(\bar{q}(0)) \cdot \frac{d\bar{q}}{d\lambda}(0) = 0$$



Note: the above definition works for any level set: if $\bar{r} \in \alpha_c$, the normal of the c-level-set at point $\bar{r}$ is given by $\nabla f(\bar{r})$

Surface Normals from the Implicit Function

Proof: Let $\bar{q}(\lambda) = (x(\lambda), y(\lambda), z(\lambda))$ be a curve on the surface α_c with $\bar{q}(0) = \bar{r}$.

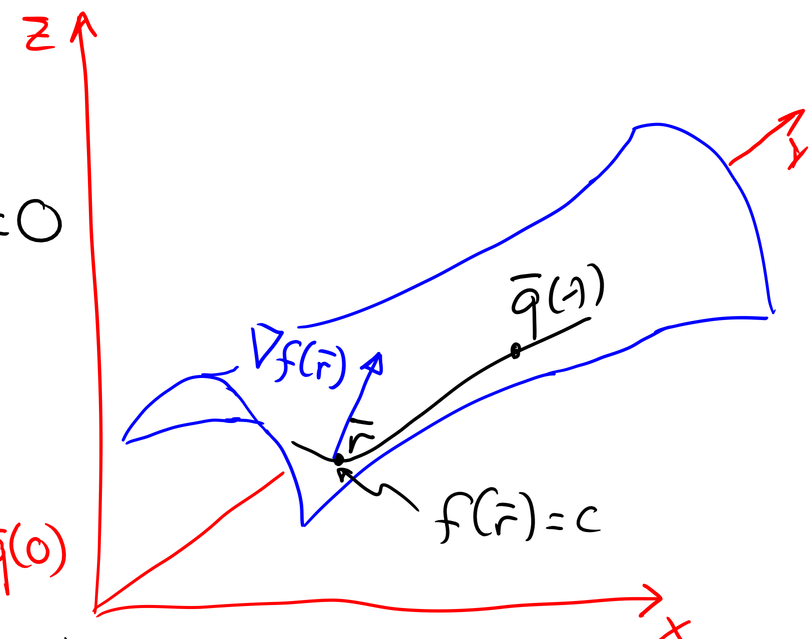
$$\Longleftrightarrow \quad \forall \lambda \quad f(\bar{q}(\lambda)) = c$$

$$\Longleftrightarrow \quad \frac{df}{d\lambda}(\bar{q}(0)) = 0$$

$$\Longleftrightarrow \quad \frac{\partial f}{\partial x} \frac{dx}{d\lambda} + \frac{\partial f}{\partial y} \cdot \frac{dy}{d\lambda} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial \lambda} = 0$$

$$\Longleftrightarrow \quad \underbrace{\nabla f(\bar{q}(0))}_{\text{gradient at } \bar{r}} \cdot \underbrace{\frac{d\bar{q}}{d\lambda}(0)}_{\text{3D tangent at } \bar{q}(0)} = 0$$

gradient at $\bar{r}$ — 3D tangent at $\bar{q}(0)$



since the above orthogonality holds for any curve in α_c through $\bar{r}$, the gradient must be perpendicular to the tangent plane

QED

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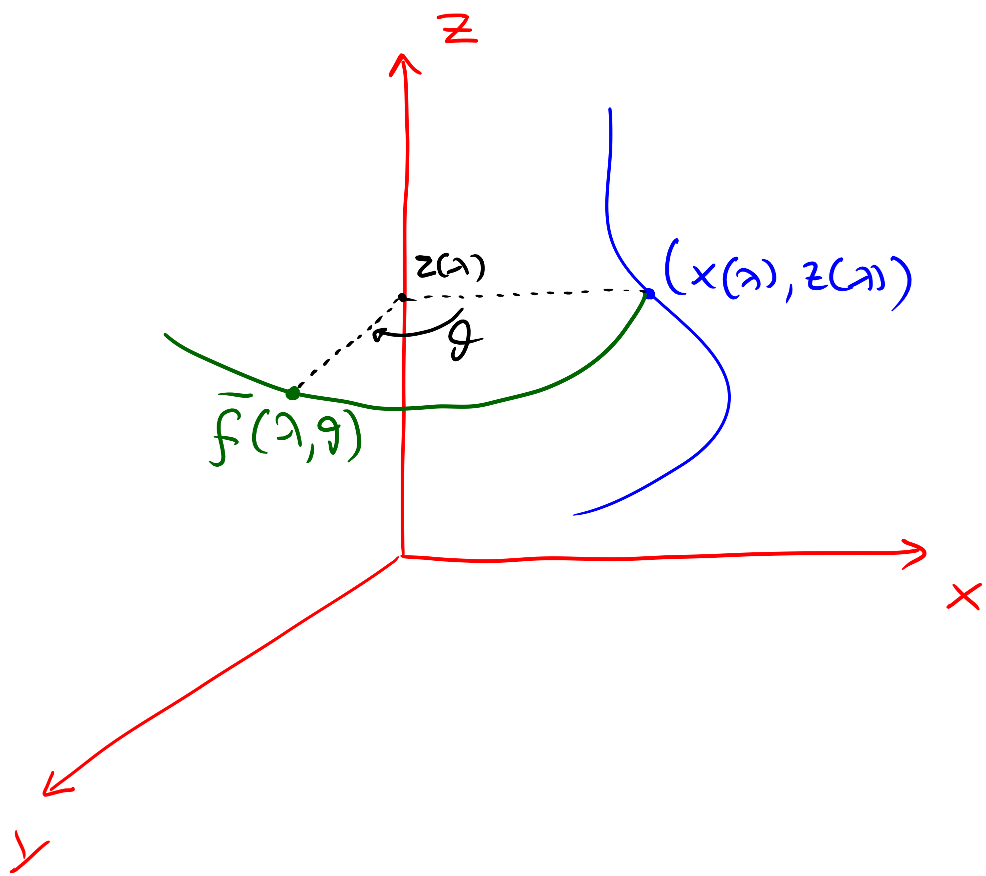
3D parametric surfaces

- Extrude
- Revolve
- Loft
- Square

Surfaces of Revolution: Basic Construction



Conceptual steps:

1. Define a 2D curve on the xz-plane:
2. Rotate it about the z-axis

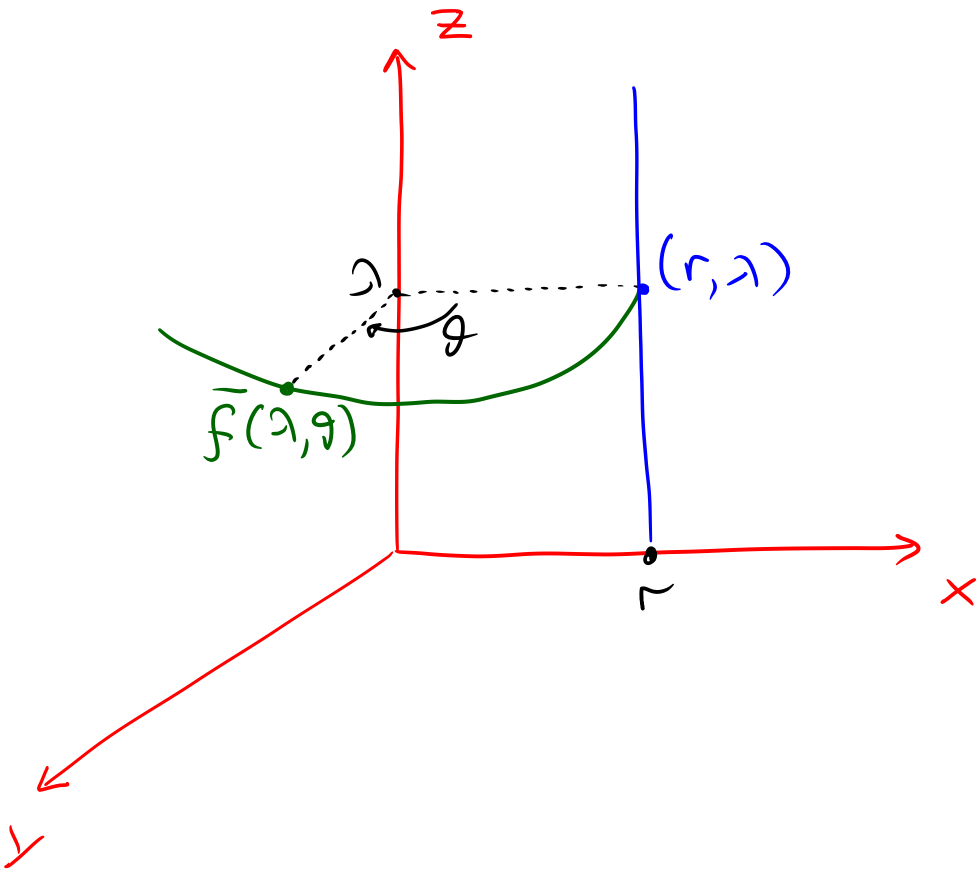
Another equivalent view:

- Point (x(λ), y(λ)) will trace a circle in xy-plane
- The circle's radius is equal to x(λ)
- θ ranges in [0, 2π)

Parametric representation

$$\bar{f}(\lambda,\theta) = \left(x(\lambda)\cos\theta,\ x(\lambda)\sin\theta,\ z(\lambda) \right)$$

Example: The Cylinder



Question: how do we express the cylinder of radius r ?

Ans:

$$(x(\lambda), z(\lambda)) = (r, \lambda)$$

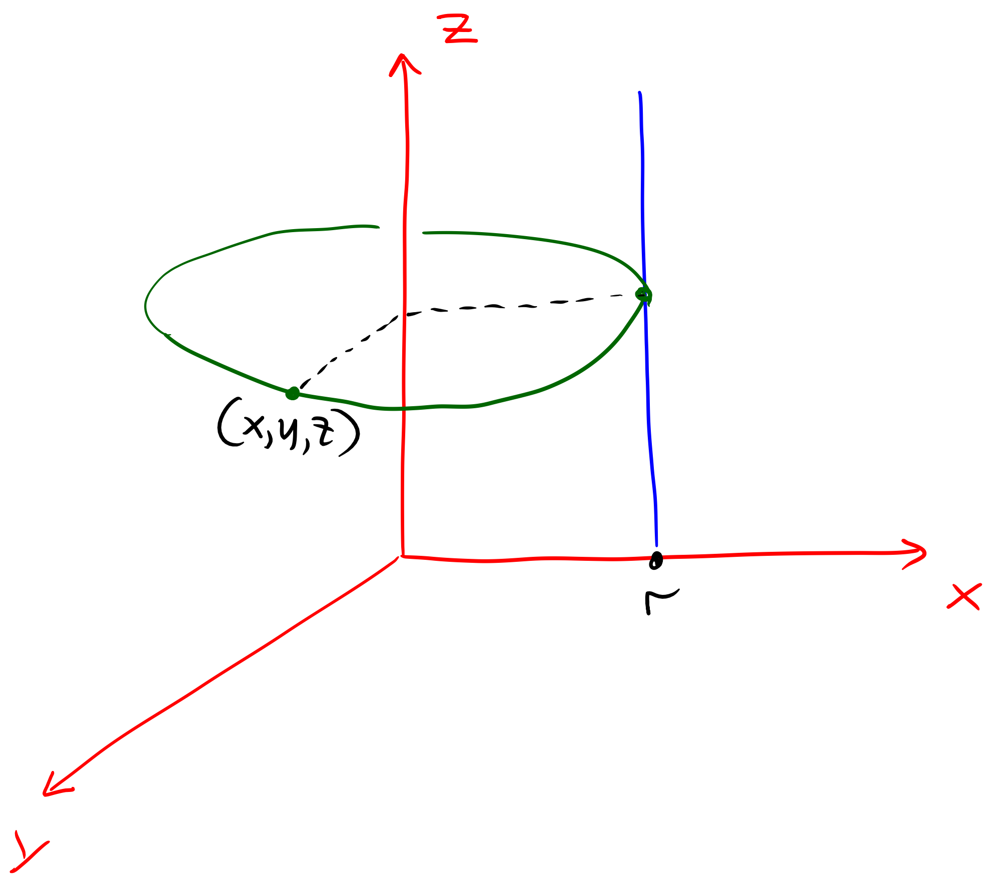
So

$$\bar{f}(\lambda,\theta) = (r\cos\theta, r\sin\theta, \lambda)$$

Parametric representation

$$\bar{f}(\lambda,\theta) = (x(\lambda)\cos\theta, x(\lambda)\sin\theta, z(\lambda))$$

Example: Implicit Function of the Cylinder



Question: how do we express the cylinder of radius r ?

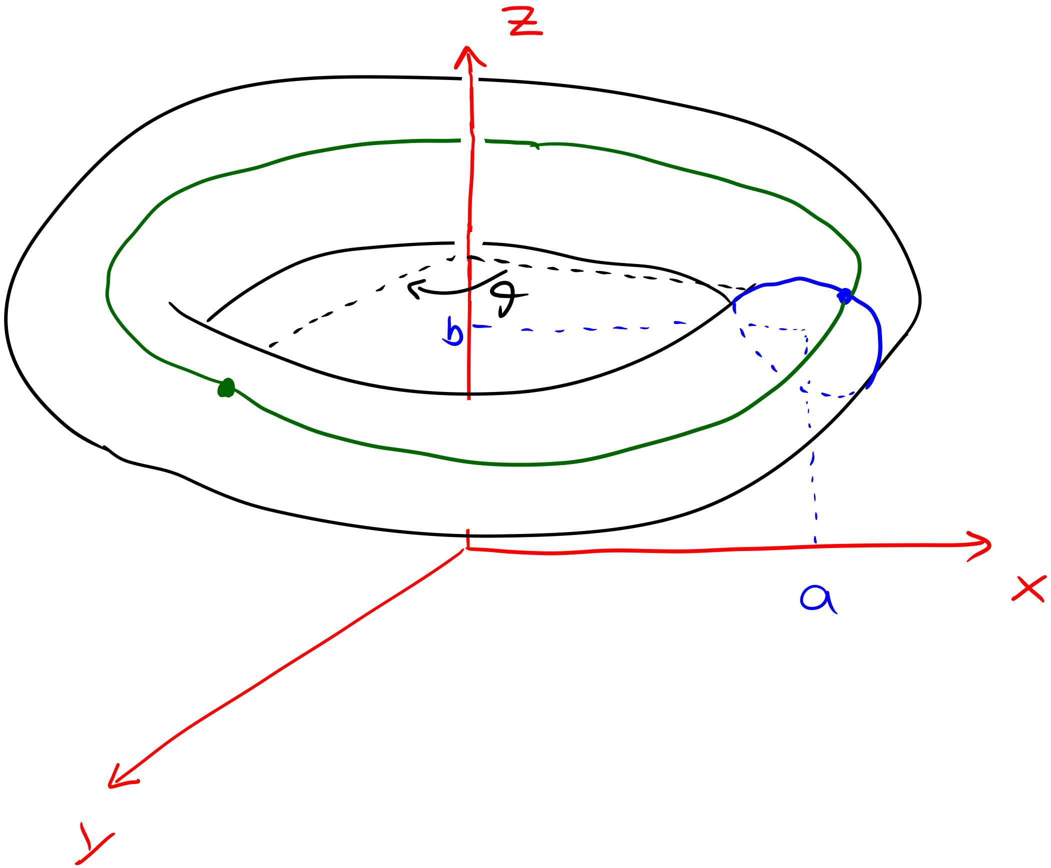
Ans:

The points (x,y,z) on the cylinder have constant distance r from z-axis

Implicit equation

$$f(x,y,z) = x^2 + y^2 - r^2 = 0$$

Example: The Torus as a Surface of Revolution



Question: how do we express the torus as a surface of revolution?

Ans: torus is formed by rotating a circle about the z-axis

$$(x(\lambda), z(\lambda)) = (r\cos\lambda + a, r\sin\lambda + b)$$

Parametric representation

$$\bar{f}(\lambda,\theta) = (x(\lambda)\cos\theta, x(\lambda)\sin\theta, z(\lambda))$$

3D parametric surfaces: Coons interpolation