VECTOR GEOMETRY:
A COORDINATE-FREE APPROACH

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Control Data Corporation

Abandon All Coordinates
Ye Who Enter Here

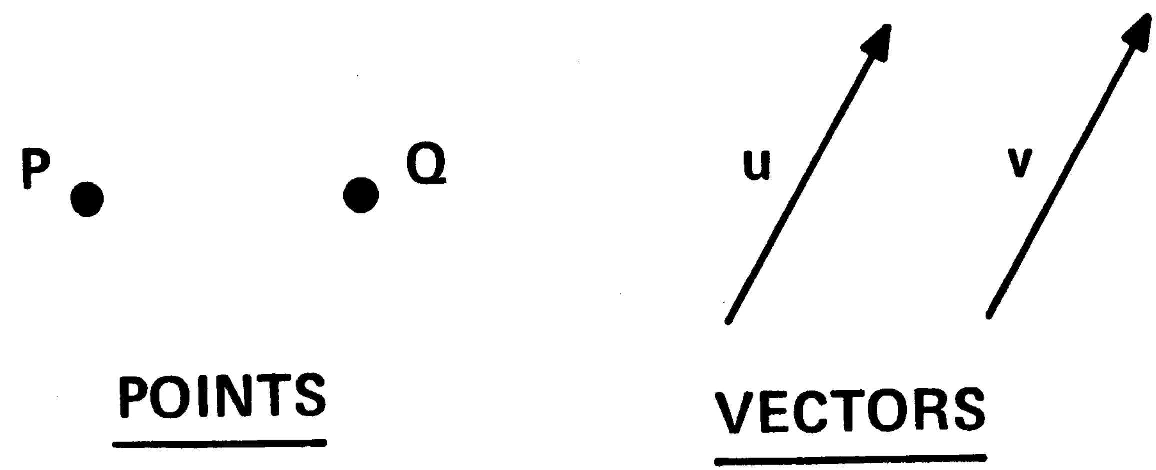
Contents

I. The Algebra of Points and Vectors

Undefined Terms

1. Position
2. Direction
3. Length
4. Point
5. Vector

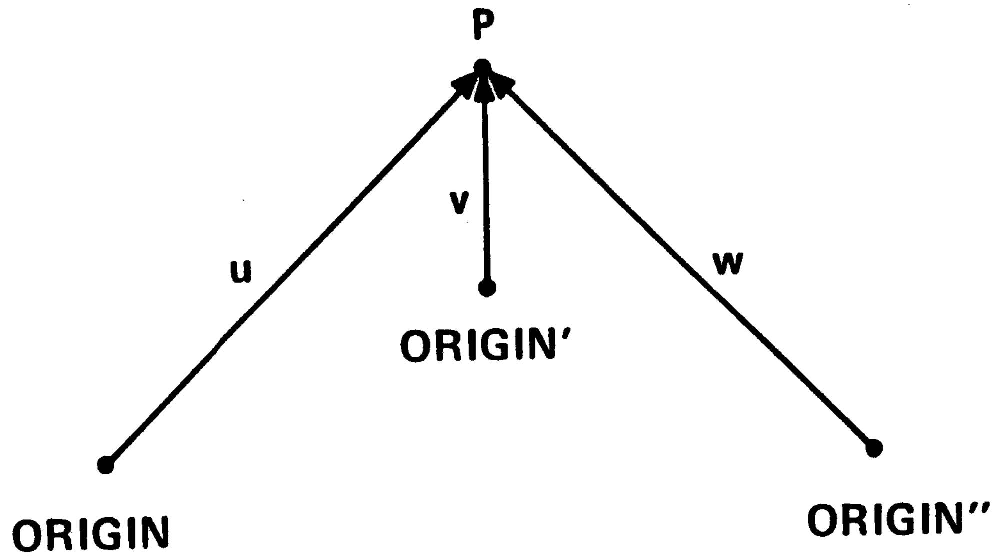
POINTS AND VECTORS



1. A POINT HAS POSITION, BUT NOT DIRECTION OR LENGTH.

2. A VECTOR HAS DIRECTION AND LENGTH, BUT NOT POSITION.

COORDINATE SYSTEMS

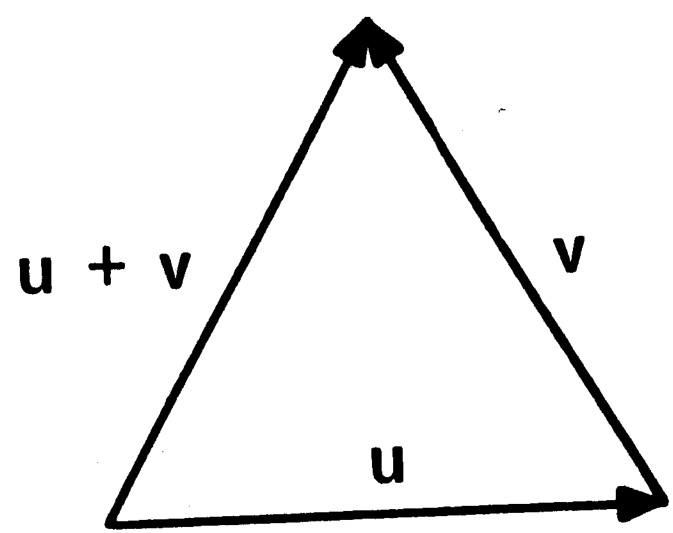


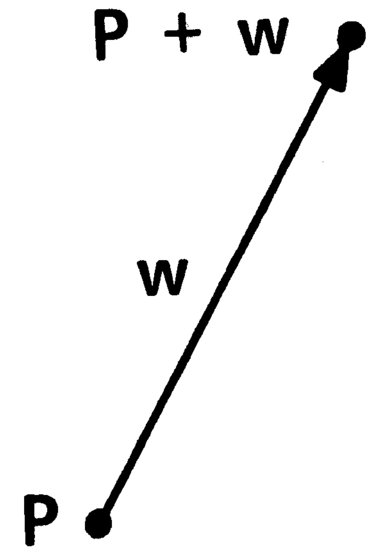
1. A POINT IS NOT A VECTOR.
2. IF AN ORIGIN IS SPECIFIED, THEN A POINT CAN BE REPRESENTED BY A VECTOR FROM THE ORIGIN TO THE POINT. HOWEVER THIS VECTOR DEPENDS ON THE PARTICULAR CHOICE OF ORIGIN. DIFFERENT ORIGINS GENERATE DIFFERENT VECTORS.
3. A POINT IS NOT A VECTOR. POINTS AND VECTORS ARE DISTINCT, COORDINATE FREE, CONCEPTS.

Vector Algebra (Coordinate Free)

1. Addition
2. Subtraction
3. Scalar Multiplication
4. Dot Product (Length)
5. Cross Product (Area)
6. Determinant (Volume)

1. ADDITION

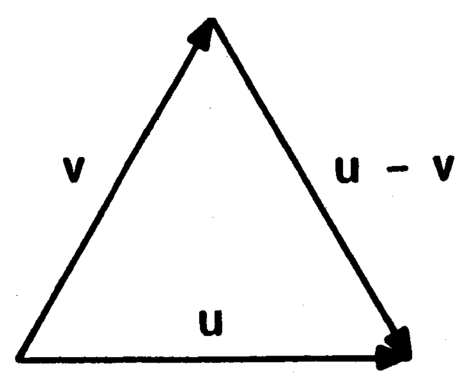




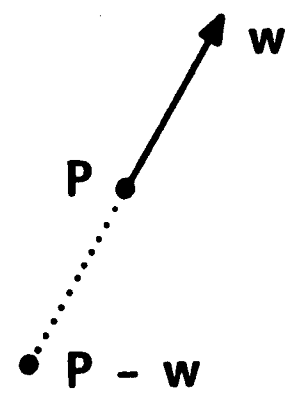
u + v IS A VECTOR

P + w IS A POINT

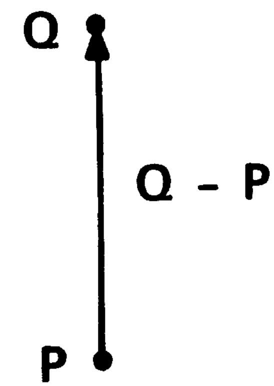
2. SUBTRACTION



u - v IS A VECTOR

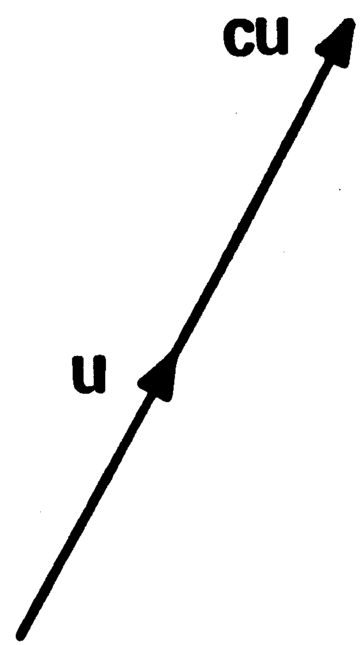


P - w IS A POINT

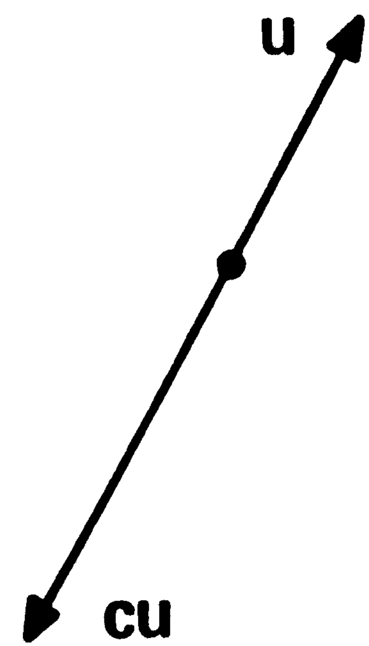


Q - P IS A VECTOR

3. SCALAR MULTIPLICATION



$c > 0$



$c < 0$

cu IS A VECTOR

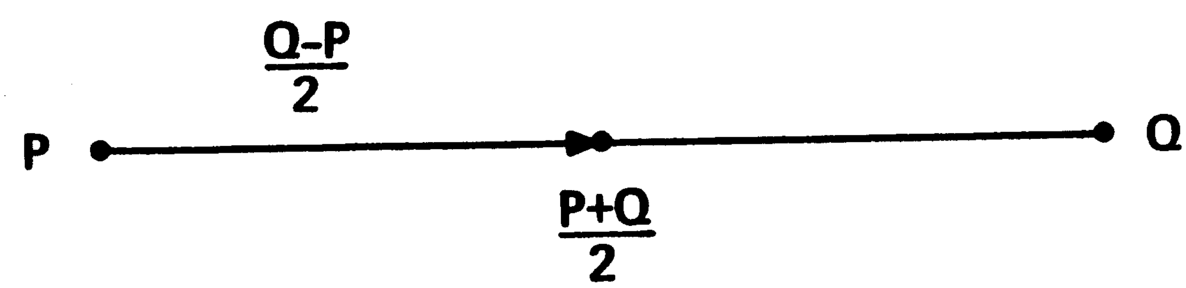
$|cu| = |c|\ |u|$

OBSERVATIONS

1. ADDITION AND SCALAR MULTIPLICATION APPLY ONLY TO VECTORS, NOT TO POINTS.

2. HOWEVER, SOMETIMES EXPRESSIONS INVOLVING ADDITION AND SCALAR MULTIPLICATION OF POINTS ARE LEGAL. FOR EXAMPLE:

$$1/2\, P + 1/2\, Q = \text{MIDPOINT OF LINE JOINING } P, Q$$



$$\frac{P+Q}{2} = P + \frac{Q-P}{2}$$

POINT = POINT + VECTOR

SCALAR MULTIPLICATION FOR POINTS

$1 \cdot P = P$	(POINT)
$0 \cdot P = 0$	(VECTOR)
$c \cdot P =$ UNDEFINED	$c \neq 0, 1$

FORMAL LINEAR COMBINATION OF POINTS

$$\sum_{K=0}^{N} c_K P_K = \underbrace{\left(\sum_{K=0}^{N} c_K\right)}_{\substack{\text{MUST} \\ \text{BE 0} \\ \text{OR 1}}} P_0 + \underbrace{\sum_{K=1}^{N} c_K (P_K - P_0)}_{\text{VECTOR}}$$

$$\sum c_K P_K = P_0 + \sum c_K (P_K - P_0) \qquad \sum c_K = 1 \quad \text{(POINT)}$$

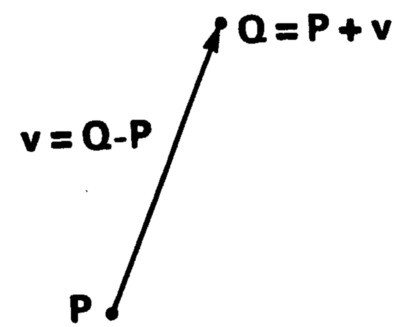
$$= \sum c_K (P_K - P_0) \qquad \sum c_K = 0 \quad \text{(VECTOR)}$$

$$= \text{UNDEFINED} \qquad \sum c_K \neq 0, 1$$

EXAMPLES

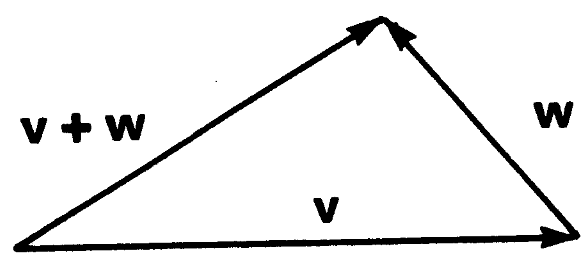
1. $\frac{P+Q}{2}$ = MIDPOINT OF LINE PQ
2. $\frac{P+Q+R}{3}$ = CENTER OF GRAVITY OF $\triangle$ PQR
3. $Q-P$ = VECTOR FROM P TO Q
4. $P+Q$ = UNDEFINED EXPRESSION

The Properties of Addition, Subtraction, and Scalar Multiplication of Points and Vectors



1.	$P + 0$	$= P$	
2.	$P - P$	$= 0$	
3.	$P - Q$	$= -(Q - P)$	
4.	$P + (v + w)$	$= (P + v) + w$	
5.	$P + v$	$= P + w$	iff $v = w$
6.	$P + v$	$= Q + v$	iff $P = Q$
7.	$(R - P)$	$= (R - Q) + (Q - P)$	
8.	$P + (Q - R)$	$= Q + (P - R)$	
9.	$v + (P - Q)$	$= (P + v) - Q$	
10.	$\Sigma c_k P_k$	$= P_0 + \Sigma c_k (P_k - P_0)$	iff $\Sigma c_k = 1$
		$= \Sigma c_k (P_k - P_0)$	iff $\Sigma c_k = 0$
		= undefined	otherwise

Addition, Subtraction, and Scalar Multiplication of Vectors

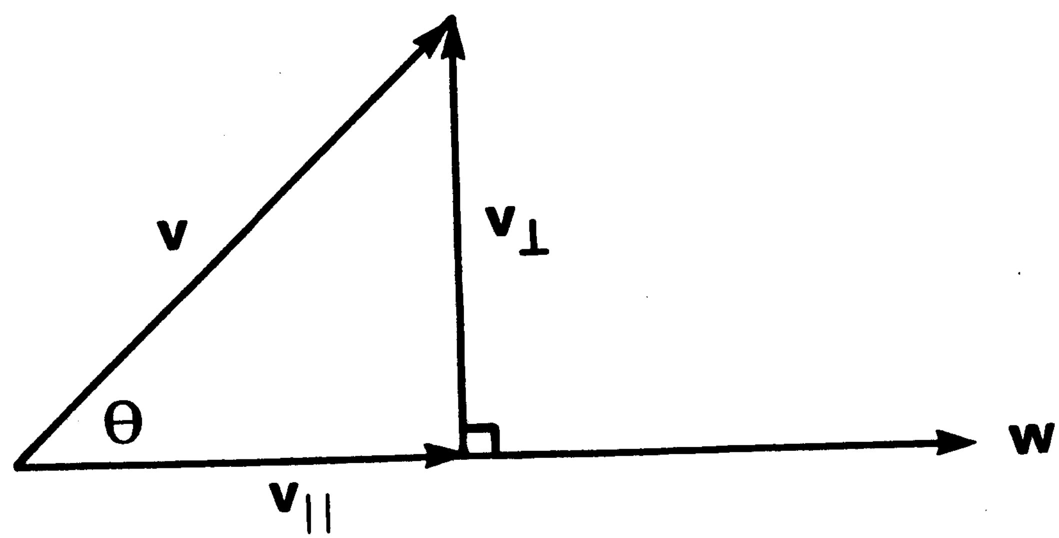


1. $v + 0 = v$
2. $v - v = 0$
3. $u + (v + w) = (u + v) + w$
4. $v + w = w + v$
5. $(c_1 + c_2)v = c_1 v + c_2 v$
6. $c(v + w) = cv + cw$
7. $c_1(c_2 v) = (c_1 c_2)v$
8. $u + v = u + w$ iff $v = w$
9. $|v + w| \leq |v| + |w|$ (Triangular Inequality)

Dimensional Analysis

1.	Point + Point	= Undefined	
2.	Point - Point	= Vector	
3.	Point ± Vector	= Point	
4.	Vector ± Vector	= Vector	
5.	Scalar · Vector	= Vector	
6.	ΣScalar · Vector	= Vector	
7.	Scalar · Point	= Point	iff Scalar = 1
		= Vector	iff Scalar = 0
		= Undefined	otherwise
8.	ΣScalar · Point	= Point	iff ΣScalar = 1
		= Vector	iff ΣScalar = 0
		= Undefined	otherwise

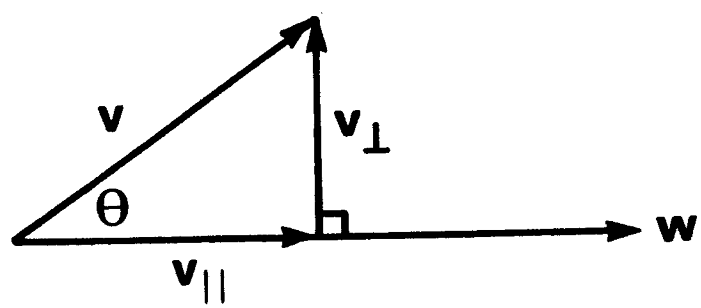
4. Dot Product (Length)



$\mathbf{v} \cdot \mathbf{w}$ is a scalar

$\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}| \; |\mathbf{w}| \cos \theta$

The Properties of the Dot Product



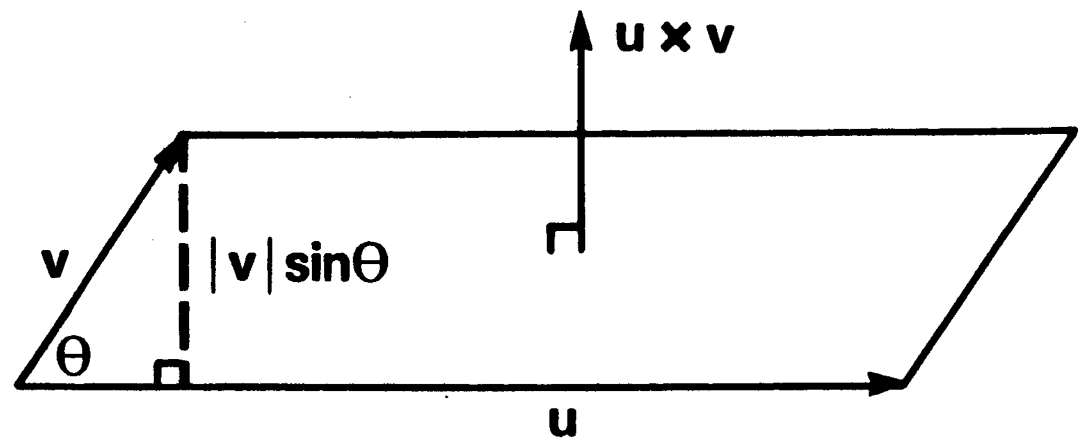
1. $\mathbf{w} \cdot \mathbf{v} = |\mathbf{w}|\ |\mathbf{v}| \cos\theta$ (Definition)
2. $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$ (Commutativity)
3. $\mathbf{w} \cdot (\mathbf{u} + \mathbf{v}) = \mathbf{w} \cdot \mathbf{u} + \mathbf{w} \cdot \mathbf{v}$
4. $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$ (Distributive Laws)
5. $(c\mathbf{w}) \cdot \mathbf{v} = c(\mathbf{w} \cdot \mathbf{v}) = \mathbf{w} \cdot (c\mathbf{v})$

The Properties of the Dot Product (Cont)

6. $|v|^2 = v \cdot v$ (Length)
7. $\cos \Theta = \frac{w \cdot v}{|w|\ |v|}$ (Angle)
8. $v_{||} = \frac{(v \cdot w)\, w}{(w \cdot w)}$ (Projection)
9. $v_{\perp} = v - \frac{(v \cdot w) w}{(w \cdot w)}$ (Normal)
10. $v \cdot w = 0$ iff $v \perp w$
11. e_1, e_2, e_3 is an orthonormal basis iff

$$e_i \cdot e_j = 1 \quad i = j$$
$$= 0 \quad i \neq j$$

5. Cross Product (Area)



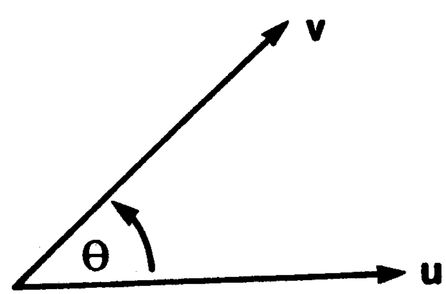
$u \times v$ is a vector

$|u \times v| = \text{Area}(u,v) = |u|\ |v| \sin\theta$

$\text{Dir}(u \times v) \perp u,v$

$\text{sgn}(u,v,u \times v) = +1$ (right hand rule)

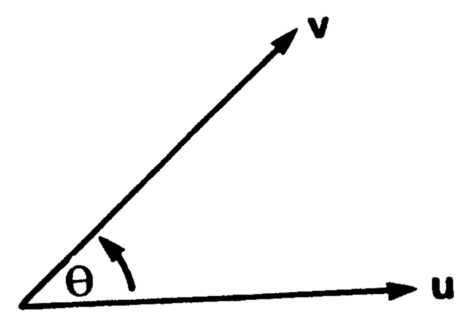
Orientation



w out of screen

$\theta > 0$

$\text{sgn}(u,v,w) = \text{sgn}(\theta) = +1$



w into screen

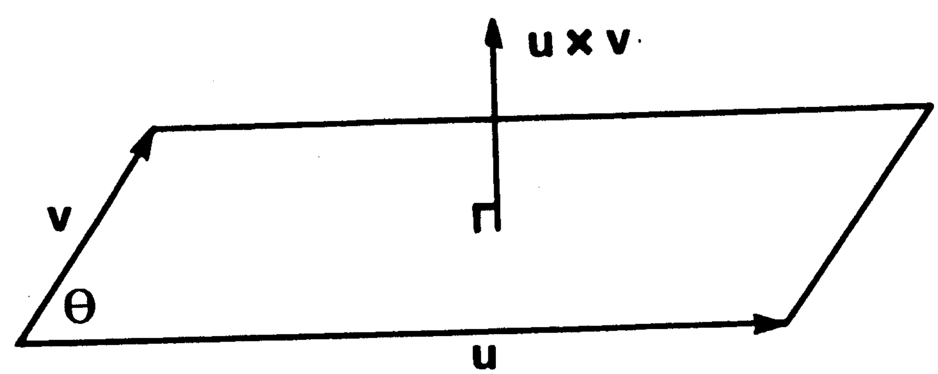
$\theta < 0$

$\text{sgn}(u,v,w) = \text{sgn}(\theta) = -1$

$$\text{sgn}(u,v,w) = \text{sgn}(v,w,u) = \text{sgn}(w,u,v)$$

$$\text{sgn}(u,v,w) = -\text{sgn}(v,u,w)$$

The Properties of the Cross Product

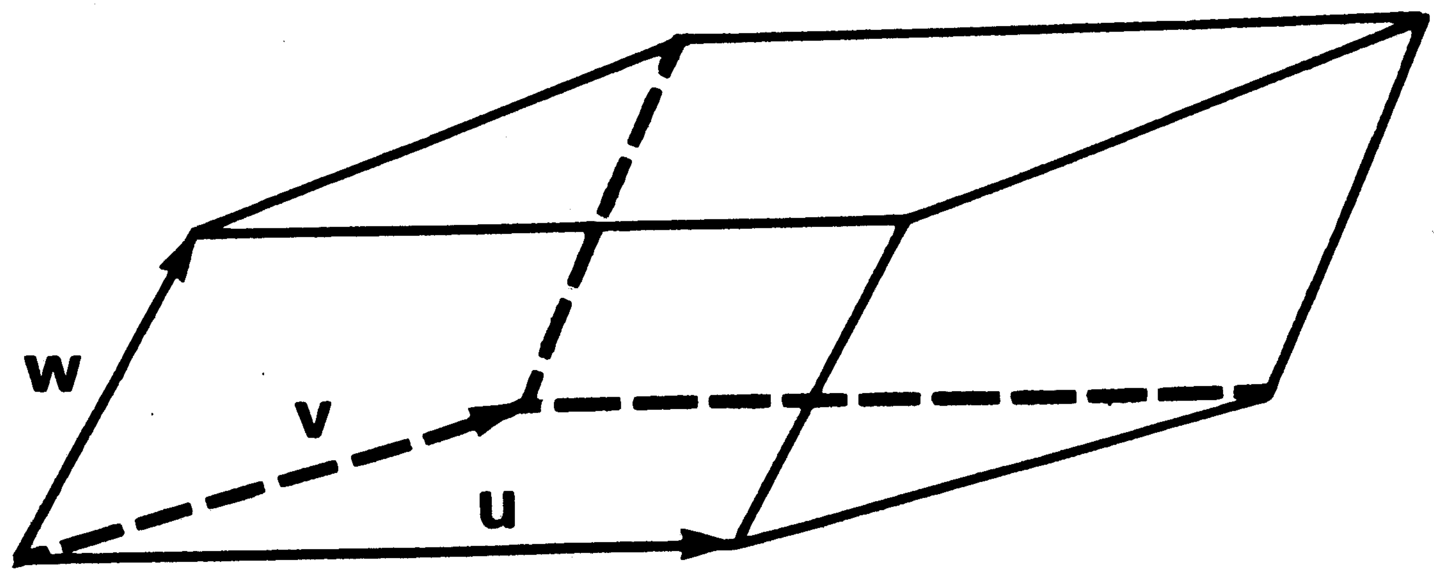


1. $|u \times v| = |u|\ |v| \sin\theta$
2. $u \times v \perp u, v$
3. $\mathrm{sgn}(u, v, u \times v) = +1$
4. $\mathrm{Area}(u, v) = |u \times v|$
5. $(u \times v) \cdot u = 0$
6. $(u \times v) \cdot v = 0$
7. $u \times v = 0$ iff $v \parallel \pm u$
8. $u \times u = 0$

The Properties of the Cross Product (Cont)

9. $u \times v = -v \times u$ (Anti-Commutativity)
10. $u \times (v + w) = u \times v + u \times w$ (Distributive Laws)
11. $(v + w) \times u = v \times u + w \times u$
12. $(cu) \times v = c(u \times v) = u \times (cv)$
13. $u \times (v \times w) = (u \cdot w)v - (u \cdot v)w$ (Non-Associativity)
14. $(u \times v) \times w = (u \cdot w)v - (v \cdot w)u$
15. $|u \times v|^2 = |u|^2 |v|^2 - (u \cdot v)^2$
16. $(u_1 \times u_2) \cdot (v_1 \times v_2) = (u_1 \cdot v_1)(u_2 \cdot v_2) - (u_1 \cdot v_2)(u_2 \cdot v_1)$ (Lagrange's Identity)
17. $e_1, e_2, e_3 \neq 0$ is an orthonormal basis with positive orientation iff
    $e_1 \times e_2 = e_3,\ e_2 \times e_3 = e_1,\ e_3 \times e_1 = e_2$

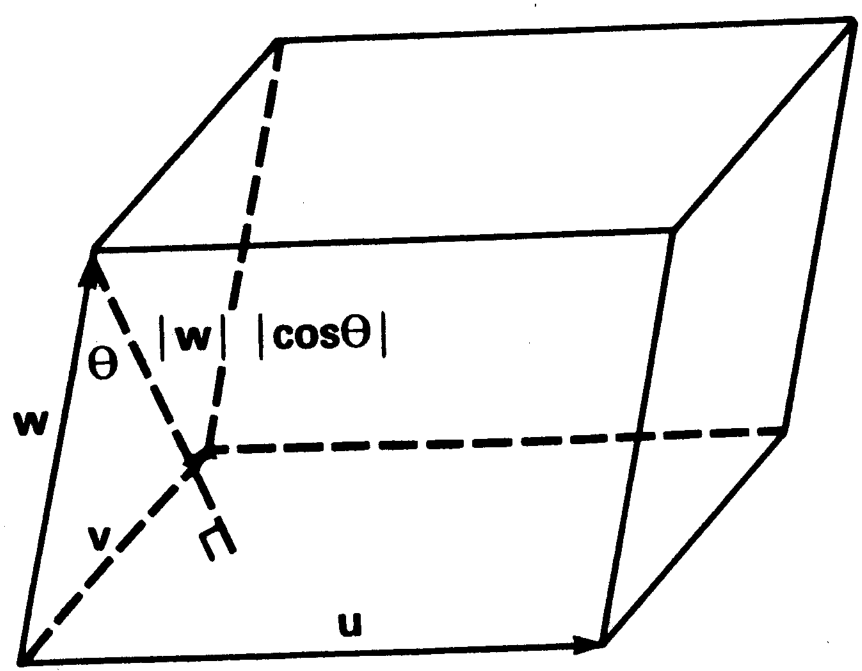
6. Determinant (Volume)



Det(u,v,w) is a scalar

Det(u,v,w) = sgn(u,v,w) Vol(u,v,w)

Scalar Triple Product



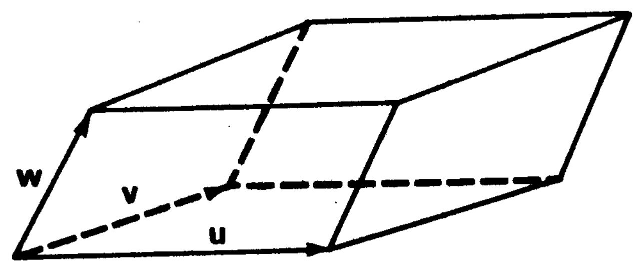
$$\mathrm{Det}(u,v,w) = (u \times v)\cdot w$$

Proof:

$$\begin{aligned}\mathrm{Det}(u,v,w) &= \mathrm{sgn}(u,v,w)\ \mathrm{Vol}(u,v,w)\\ &= \mathrm{sgn}(u,v,w)\ \mathrm{Area}(u,v)\ \mathrm{Height}(u,v,w)\\ &= \mathrm{sgn}(u,v,w)\ |u \times v|\ |w|\ |\cos\Theta|\\ &= (u \times v)\cdot w\end{aligned}$$

QED

The Properties of the Determinant Function



1. $\mathrm{Det}(u,v,w) = (u \times v) \cdot w$
2. $\mathrm{Vol}(u,v,w) = |\mathrm{Det}(u,v,w)|$
3. $\mathrm{Det}(u,v,w) \neq 0$ iff u,v,w is a basis
4. $\mathrm{Det}(u,v,w) > 0$ iff $\mathrm{sgn}(u,v,w) > 0$
5. $\mathrm{Det}(u,u,v) = \mathrm{Det}(u,v,u) = \mathrm{Det}(u,v,v) = 0$
6. $\mathrm{Det}(u,v,w) = \mathrm{Det}(v,w,u) = \mathrm{Det}(w,u,v)$
   $= -\mathrm{Det}(u,w,v) = -\mathrm{Det}(w,v,u)$
   $= -\mathrm{Det}(v,u,w)$
7. (a) $\mathrm{Det}(u_1 + u_2,v,w) = \mathrm{Det}(u_1,v,w) + \mathrm{Det}(u_2,v,w)$
   (b) $\mathrm{Det}(u,v_1 + v_2 w) = \mathrm{Det}(u,v_1,w) + \mathrm{Det}(u,v_2,w)$
   (c) $\mathrm{Det}(u,v,w_1 + w_2) = \mathrm{Det}(u,v,w_1) + \mathrm{Det}(u,v,w_2)$
8. $\mathrm{Det}(cu,v,w) = \mathrm{Det}(u,cv,w) = \mathrm{Det}(u,v,cw) = c\mathrm{Det}(u,v,w)$
9. (a) $\mathrm{Det}(c_1 u + c_2 v + c_3 w,v,w) = c_1 \mathrm{Det}(u,v,w)$
   (b) $\mathrm{Det}(u,c_1 u + c_2 v + c_3 w,w) = c_2 \mathrm{Det}(u,v,w)$
   (c) $\mathrm{Det}(u,v,c_1 u + c_2 v + c_3 w) = c_3 \mathrm{Det}(u,v,w)$
10. If e_1,e_2,e_3 is an orthonormal basis with positive orientation, then $\mathrm{Det}(e_1,e_2,e_3) = +1$.

Summary

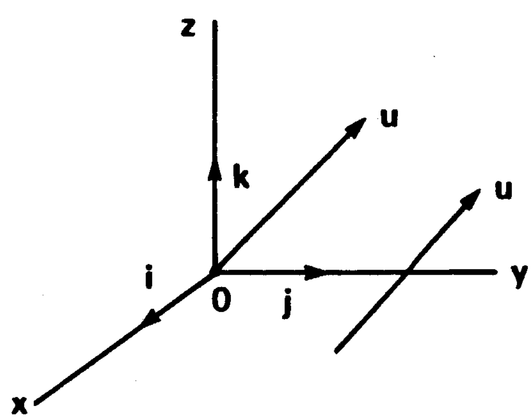
Operation	Legal	Illegal
Addition	$u + v$ (vector) $P + w$ (point)	$P + Q$
Subtraction	$u - v$ (vector) $P - w$ (point) $Q - P$ (vector)	$u - P$
Scalar Multiplication	cu (vector) $1 \cdot P$ (point) $0 \cdot P$ (vector)	cP ($c \neq 0, 1$)
Dot Product	$u \cdot v$ (scalar)	$P \cdot Q$, $P \cdot v$
Cross Product	$u \times v$ (vector)	$P \times Q$, $P \times v$
Determinant	$Det(u,v,w)$ (scalar)	$Det(P,Q,R)$, $Det(P,Q,w)$, $Det(P,v,w)$

Supplement on Coordinates

1. Addition
2. Subtraction
3. Scalar Multiplication
4. Dot Product
5. Cross Product
6. Determinant

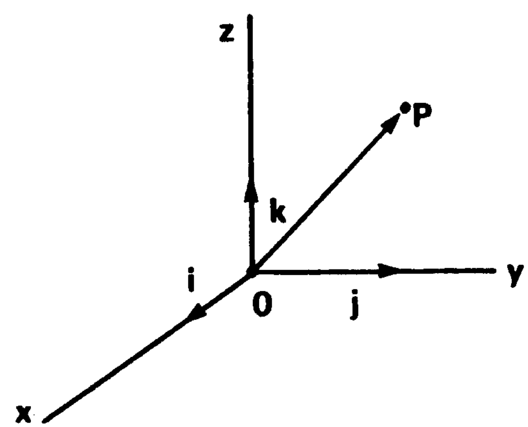
Rectangular Coordinates

1. Vectors



$u = (u_1, u_2, u_3)$ means $u = u_1 i + u_2 j + u_3 k$

2. Points



$P = (p_1, p_2, p_3)$ means $P = 0 + p_1 i + p_2 j + p_3 k$

Addition-Subtraction

1. Vector ± Vector

$$u = (u_1, u_2, u_3) = u_1 i + u_2 j + u_3 k$$

$$v = (v_1, v_2, v_3) = v_1 i + v_2 j + v_3 k$$

$$u \pm v = (u_1 i + u_2 j + u_3 k) \pm (v_1 i + v_2 j + v_3 k)$$

$$= (u_1 \pm v_1) i + (u_2 \pm v_2) j + (u_3 \pm v_3) k$$

$$= (u_1 \pm v_1,\ u_2 \pm v_2,\ u_3 \pm v_3)$$

Addition-Subtraction (Cont)

2. Point $\pm$ Vector

$$P = (p_1,p_2,p_3) \quad \text{means} \quad P = 0 + p_1 i + p_2 j + p_3 k$$

$$u = (u_1,u_2,u_3) = u_1 i + u_2 j + u_3 k$$

$$\begin{aligned} P \pm u &= 0 + (p_1 i + p_2 j + p_3 k) \pm (u_1 i + u_2 j + u_3 k) \\ &= 0 + (p_1 \pm u_1) i + (p_2 \pm u_2) j + (p_3 \pm u_3) k \\ &= (p_1 \pm u_1, p_2 \pm u_2, p_3 \pm u_3) \end{aligned}$$

Scalar Multiplication

$$u = (u_1, u_2, u_3) = u_1 i + u_2 j + u_3 k$$

$$\begin{aligned} cu &= c(u_1 i + u_2 j + u_3 k) \\ &= (cu_1)i + (cu_2)j + (cu_3)k \\ &= (cu_1, cu_2, cu_3) \end{aligned}$$

Dot Product

$$u \cdot v = |u|\ |v| \cos\theta \qquad \text{(definition)}$$

$$i \cdot i = j \cdot j = k \cdot k = 1$$

$$i \cdot j = j \cdot k = k \cdot i = 0$$

$$u = (u_1, u_2, u_3) = u_1 i + u_2 j + u_3 k$$

$$v = (v_1, v_2, v_3) = v_1 i + v_2 j + v_3 k$$

$$u \cdot v = (u_1 i + u_2 j + u_3 k) \cdot (v_1 i + v_2 j + v_3 k)$$

$$= u_1 v_1 + u_2 v_2 + u_3 v_3$$

Cross Product

$|u \times v| = |u|\ |v|\ \sin\Theta$ (definition)

Dir($u \times v$) $\perp$ u,v with orientation by right hand rule (definition)

$$i \times i = j \times j = k \times k = 0$$

$$i \times j = k,\ j \times k = i,\ k \times i = j$$

$$j \times i = -k,\ k \times j = -i,\ i \times k = -j$$

Cross Product (Cont)

$$u = (u_1, u_2, u_3) = u_1 i + u_2 j + u_3 k$$

$$v = (v_1, v_2, v_3) = v_1 i + v_2 j + v_3 k$$

$$u x v = (u_1 i + u_2 j + u_3 k) \text{ x } (v_1 i + v_2 j + v_3 k)$$

$$= (u_2 v_3 - u_3 v_2) i + (u_3 v_1 - u_1 v_3) j + (u_1 v_2 - u_2 v_1) k$$

$$= (u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1)$$

$$= \text{Det} \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Determinant

$$u = (u_1, u_2, u_3)$$

$$v = (v_1, v_2, v_3)$$

$$w = (w_1, w_2, w_3)$$

$$\begin{aligned}
\text{Det}\,(u,v,w) &= (u \times v) \cdot w \\
&= (u_2v_3 - u_3v_2,\ u_3v_1 - u_1v_3,\ u_1v_2 - u_2v_1) \cdot (w_1, w_2, w_3) \\
&= (u_2v_3 - u_3v_2)w_1 + (u_3v_1 - u_1v_3)w_2 + (u_1v_2 - u_2v_1)w_3 \\
&= \text{Det} \begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix}
\end{aligned}$$

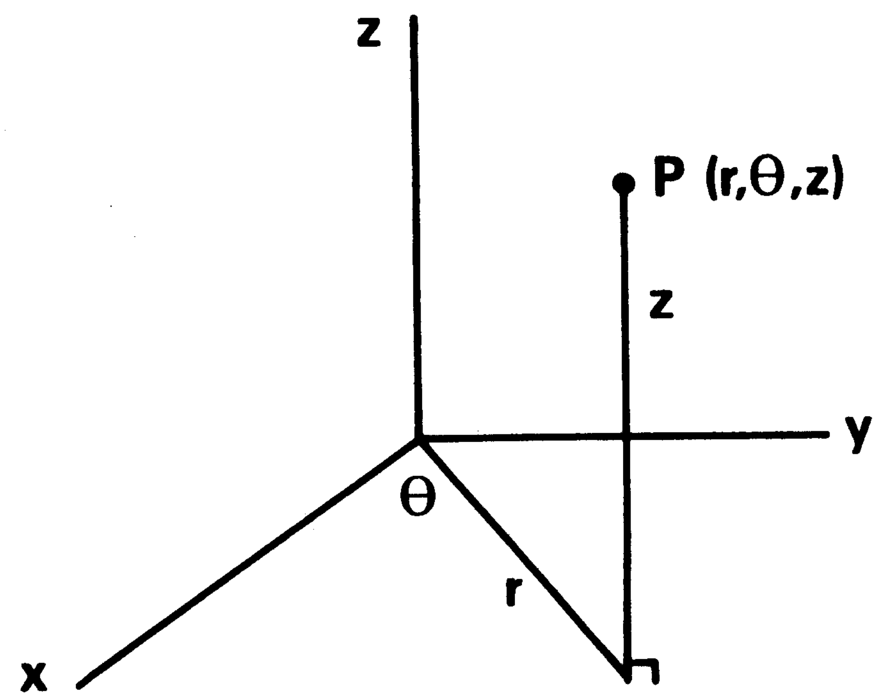
Exercise

Find the formulas for addition, subtraction, scalar multiplication, dot product, cross product, and the determinant function in terms of:

a. cylindrical coordinates

b. spherical coordinates

Cylindrical Coordinates



$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

Dot Product

Vector	Rectangular Coordinates	Cylindrical Coordinates
u	(x_u, y_u, z_u)	(r_u, Θ_u, z_u)
v	(x_v, y_v, z_v)	(r_v, Θ_v, z_v)

$$(u \cdot v)_{rectangular} = x_u x_v + y_u y_v + z_u z_v$$

$$(u \cdot v)_{cylindrical} = r_u r_v \cos(\Theta_u - \Theta_v) + z_u z_v$$