# 14 Interpolation

### **14.1** Interpolation Basics

*Goal:* We would like to be able to define curves in a way that meets the following criteria:

- 1. Interaction should be natural and intuitive.
- 2. Smoothness should be controllable.
- 3. Analytic derivatives should exist and be easy to compute.
- 4. Representation should be compact.

**Interpolation** is when a curve passes through a set of "control points."



Figure 1: \*
Interpolation

**Approximation** is when a curve approximates but doesn't necessarily contain its control points.



Figure 2: \*
Approximation

**Extrapolation** is extending a curve beyond the domain of its control points.

**Continuity** - A curve is is  $C^n$  when it is continuous in up to its  $n^{th}$ -order derivatives. For example, a curve is in  $C^1$  if it is continuous and its first derivative is also continuous.

Consider a cubic interpolant — a 2D curve,  $\bar{c}(t) = \begin{bmatrix} x(t) & y(t) \end{bmatrix}$  where

$$x(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3, (1)$$

$$y(t) = b_0 + b_1 t + b_2 t^2 + b_3 t^3, (2)$$



Figure 3: \* Extrapolation

SO

$$x(t) = \sum_{i=0}^{3} a_i t^i = \begin{bmatrix} 1 & t & t^2 & t^3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \vec{t}^T \vec{a}.$$
 (3)

Here,  $\vec{t}$  is the basis and  $\vec{a}$  is the coefficient vector. Hence,  $\vec{c}(t) = \vec{t}^T \begin{bmatrix} \vec{a} & \vec{b} \end{bmatrix}$ . (Note:  $T \begin{bmatrix} \vec{a} & \vec{b} \end{bmatrix}$  is a  $4 \times 2$  matrix).

There are eight unknowns, four  $a_i$  values and four  $b_i$  values. The constraints are the values of  $\bar{c}(t)$ at known values of t.

#### Example:

For  $t \in (0,1)$ , suppose we know  $\bar{c}_j \equiv \bar{c}(t_j)$  for  $t_j = 0, \frac{1}{3}, \frac{2}{3}, 1$  as j = 1, 2, 3, 4. That is,

$$\bar{c}_1 = \begin{bmatrix} x_1 & y_1 \end{bmatrix} \equiv \begin{bmatrix} x(0) & y(0) \end{bmatrix}, \tag{4}$$

$$\bar{c}_{1} = \begin{bmatrix} x_{1} & y_{1} \end{bmatrix} \equiv \begin{bmatrix} x(0) & y(0) \end{bmatrix}, \tag{4} 
\bar{c}_{2} = \begin{bmatrix} x_{2} & y_{2} \end{bmatrix} \equiv \begin{bmatrix} x(1/3) & y(1/3) \end{bmatrix}, \tag{5} 
\bar{c}_{3} = \begin{bmatrix} x_{3} & y_{3} \end{bmatrix} \equiv \begin{bmatrix} x(2/3) & y(2/3) \end{bmatrix}, \tag{6} 
\bar{c}_{4} = \begin{bmatrix} x_{4} & y_{4} \end{bmatrix} \equiv \begin{bmatrix} x(1) & y(1) \end{bmatrix}. \tag{7}$$

$$\bar{c}_3 = \left[ \begin{array}{cc} x_3 & y_3 \end{array} \right] \equiv \left[ \begin{array}{cc} x(2/3) & y(2/3) \end{array} \right], \tag{6}$$

$$\bar{c}_4 = \left[ \begin{array}{ccc} x_4 & y_4 \end{array} \right] \equiv \left[ \begin{array}{ccc} x(1) & y(1) \end{array} \right]. \tag{7}$$

So we have the following linear system,

$$\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1/3 & (1/3)^2 & (1/3)^3 \\ 1 & 2/3 & (2/3)^2 & (2/3)^3 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \vec{a} & \vec{b} \end{bmatrix}, \tag{8}$$

or more compactly,  $\begin{bmatrix} \vec{x} & \vec{y} \end{bmatrix} = C \begin{bmatrix} \vec{a} & \vec{b} \end{bmatrix}$ . Then,  $\begin{bmatrix} \vec{a} & \vec{b} \end{bmatrix} = C^{-1} \begin{bmatrix} \vec{x} & \vec{y} \end{bmatrix}$ . From this we can find  $\vec{a}$  and  $\vec{b}$ , to calculate the cubic curve that passes through the given points.

We can also place derivative constraints on interpolant curves. Let

$$\vec{\tau}(t) = \frac{d\vec{c}(t)}{dt} = \frac{d}{dt} \begin{bmatrix} 1 & t & t^2 & t^3 \end{bmatrix} \begin{bmatrix} \vec{a} & \vec{b} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & t & t^2 \end{bmatrix} \begin{bmatrix} \vec{a} & \vec{b} \end{bmatrix},$$
(9)

that is, a different basis with the same coefficients.

#### Example:

Suppose we are given three points,  $t_j = 0, \frac{1}{2}, 1$ , and the derivative at a point,  $\vec{\tau}_2(\frac{1}{2})$ . So we can write this as

$$\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_2' & y_2' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1/2 & (1/2)^2 & (1/2)^3 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2(1/2) & 3(1/2)^2 \end{bmatrix} \begin{bmatrix} \vec{a} & \vec{b} \end{bmatrix}, \tag{11}$$

and

$$\begin{bmatrix} \bar{c}_1 \\ \bar{c}_2 \\ \bar{c}_3 \\ \vec{\tau}_2 \end{bmatrix} = C \begin{bmatrix} \vec{a} & \vec{b} \end{bmatrix}, \tag{12}$$

which we can use to find  $\vec{a}$  and  $\vec{b}$ :

$$\begin{bmatrix} \vec{a} & \vec{b} \end{bmatrix} = C^{-1} \begin{bmatrix} \bar{c}_1 \\ \bar{c}_2 \\ \bar{c}_3 \\ \bar{\tau}_2 \end{bmatrix}. \tag{13}$$

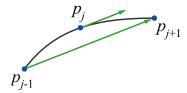
Unfortunately, polynomial interpolation yields unintuitive results when interpolating large numbers of control points; you can easily get curves that pass through the control points, but oscillate in very unexpected ways. Hence, direct polynomial interpolation is rarely used except in combination with other techniques.

## 14.2 Catmull-Rom Splines

Catmull-Rom Splines interpolate degree-3 curves with  $\mathbb{C}^1$  continuity and are made up of cubic curves.

A user specifies only the points  $[\bar{p}_1,...\bar{p}_N]$  for interpolation, and the tangent at each point is set to be parallel to the vector between adjacent points. So the tangent at  $\bar{p}_j$  is  $\kappa(\bar{p}_{j+1} - \bar{p}_{j-1})$  (for

endpoints, the tangent is instead parallel to the vector from the endpoint to its only neighbor). The value of  $\kappa$  is set by the user, determining the "tension" of the curve.



Between two points,  $\bar{p}_i$  and  $\bar{p}_{i+1}$ , we draw a cubic curve using  $\bar{p}_i$ ,  $\bar{p}_{i+1}$ , and two auxiliary points on the tangents,  $\kappa(\bar{p}_{j+1} - \bar{p}_{j-1})$  and  $\kappa(\bar{p}_{j+2} - \bar{p}_{j})$ .

We want to find the coefficients  $a_j$  when  $x(t) = \begin{bmatrix} 1 & t & t^2 & t^3 \end{bmatrix} \begin{bmatrix} a_0 & a_1 & a_2 & a_3 \end{bmatrix}^T$ , where the curve is defined as  $\bar{c}(t) = [c(t) \ y(t)]$  (similarly for y(t) and  $b_i$ ). For the curve between  $\bar{p}_i$  and  $\bar{p}_{j+1}$ , assume we know two end points,  $\bar{c}(0)$  and  $\bar{c}(1)$  and their tangents,  $\bar{c}'(0)$  and  $\bar{c}'(1)$ . That is,

$$x(0) = x_j, (14)$$

$$x(1) = x_{j+1}, (15)$$

$$x(1) = x_{j+1},$$

$$x'(0) = \kappa(x_{j+1} - x_{j-1}),$$

$$x'(1) = \kappa(x_{j+2} - x_{j}).$$
(15)
(16)

$$x'(1) = \kappa(x_{j+2} - x_j). (17)$$

To solve for  $\vec{a}$ , set up the linear system,

$$\begin{bmatrix} x(0) \\ x(1) \\ x'(0) \\ x'(1) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}.$$
(18)

Then  $\vec{x} = M\vec{a}$ , so  $\vec{a} = M^{-1}\vec{x}$ . Substituting  $\vec{a}$  in x(t) yields

$$x(t) = \begin{bmatrix} 1 & t & t^2 & t^3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 3 & -2 & -1 \\ 2 & -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_j \\ x_{j+1} \\ \kappa(x_{j+1} - x_{j-1}) \\ \kappa(x_{j+2} - x_j) \end{bmatrix}$$
(19)

$$= \begin{bmatrix} 1 & t & t^2 & t^3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\kappa & 0 & \kappa & 0 \\ 2\kappa & \kappa - 3 & 3 - 2\kappa & -\kappa \\ -\kappa & 2 - \kappa & \kappa - 2 & \kappa \end{bmatrix} \begin{bmatrix} x_{j-1} \\ x_j \\ x_{j+1} \\ x_{j+2} \end{bmatrix}.$$
 (20)

For the first tangent in the curve, we cannot use the above formula. Instead, we use:

$$\vec{\tau}_1 = \kappa(\bar{p}_2 - \bar{p}_1) \tag{21}$$

and, for the last tangent:

$$\vec{\tau}_N = \kappa(\bar{p}_N - \bar{p}_{N-1}) \tag{22}$$