Orthogonal and Symmetric Haar Wavelets on the Sphere

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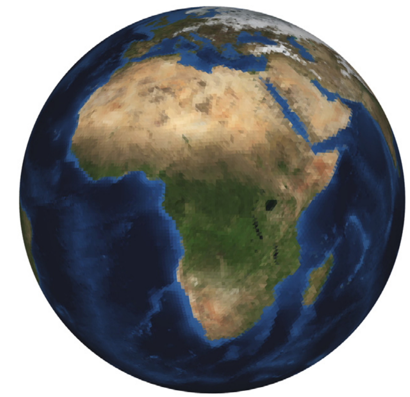
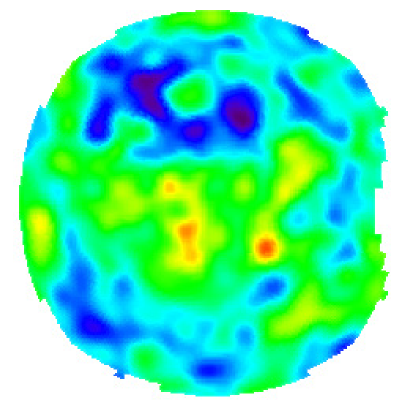
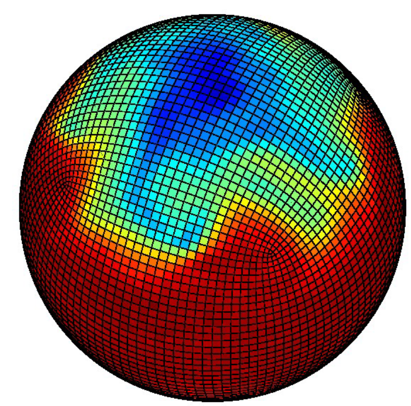
dgp
dynamic graphics project

Motivation

- Spherically parametrized signals $f \in L_2(\mathbb{S}^2, d\omega)$ exist in many fields:
  - Computer graphics,
  - Physics,
  - Astronomy,
  - Climate modeling,
  - Medical imaging,
  - ...







Objectives

- Efficient *representation* of spherical signals.
- Efficient *processing* of spherical signals.

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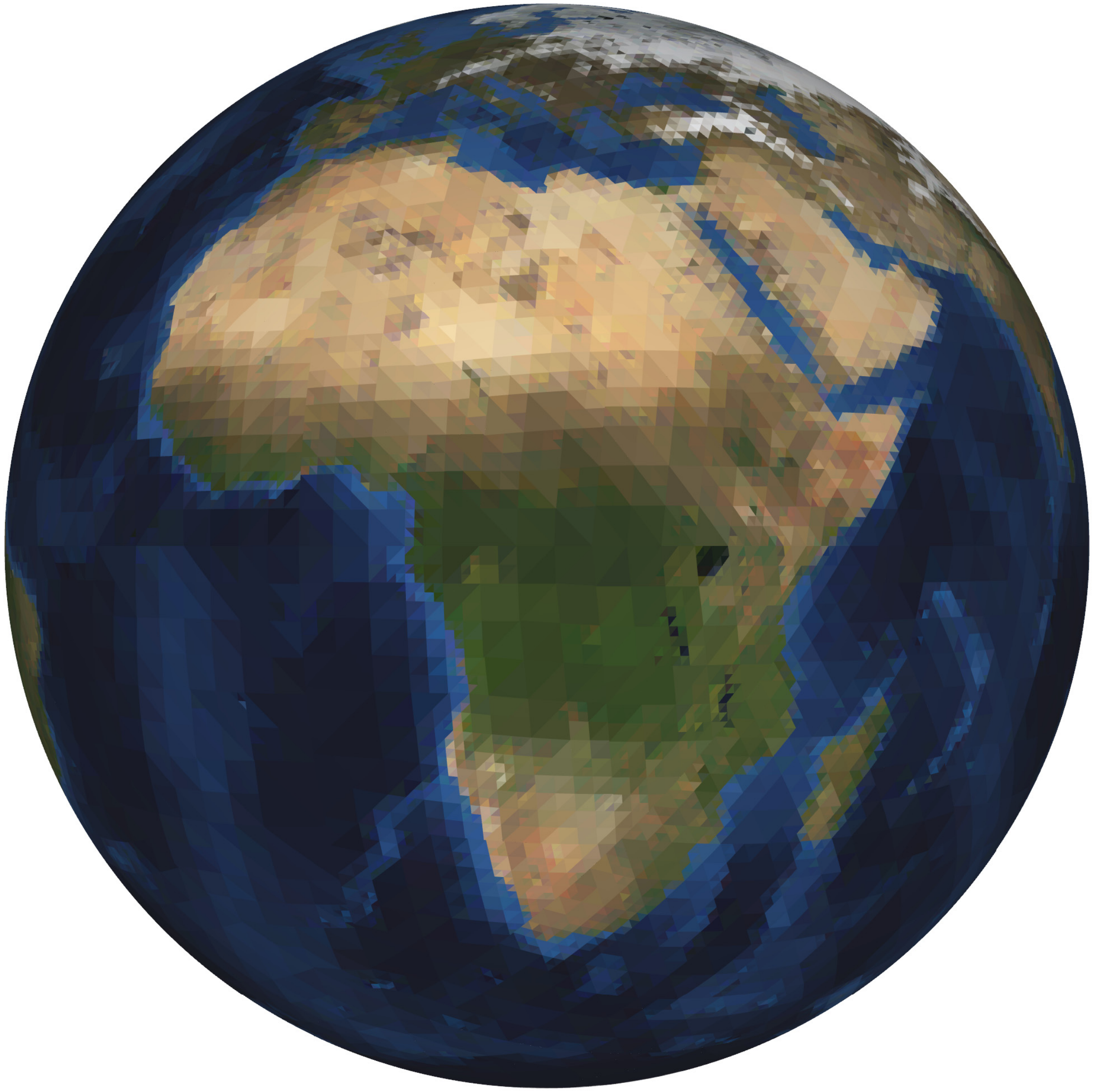

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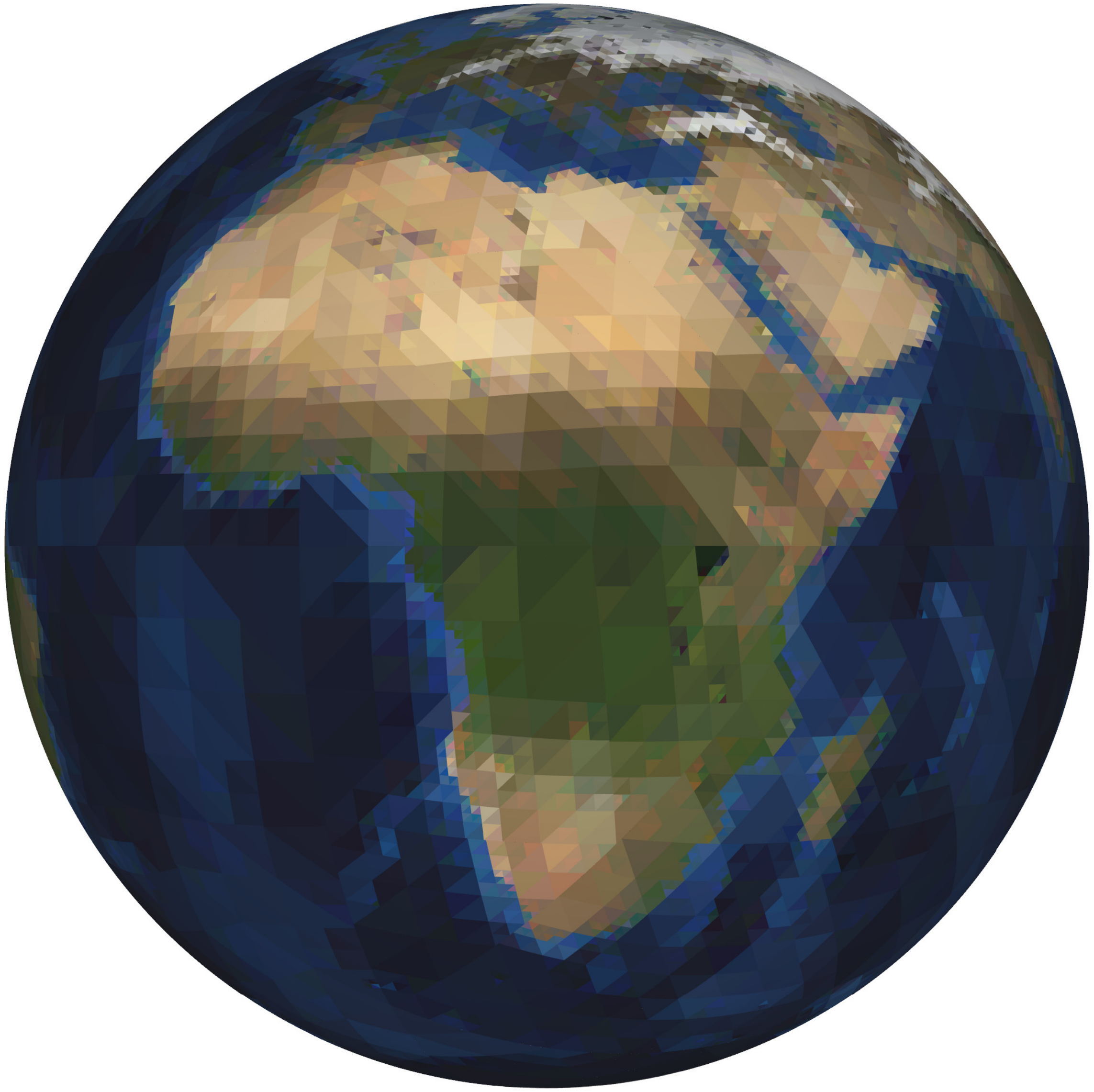

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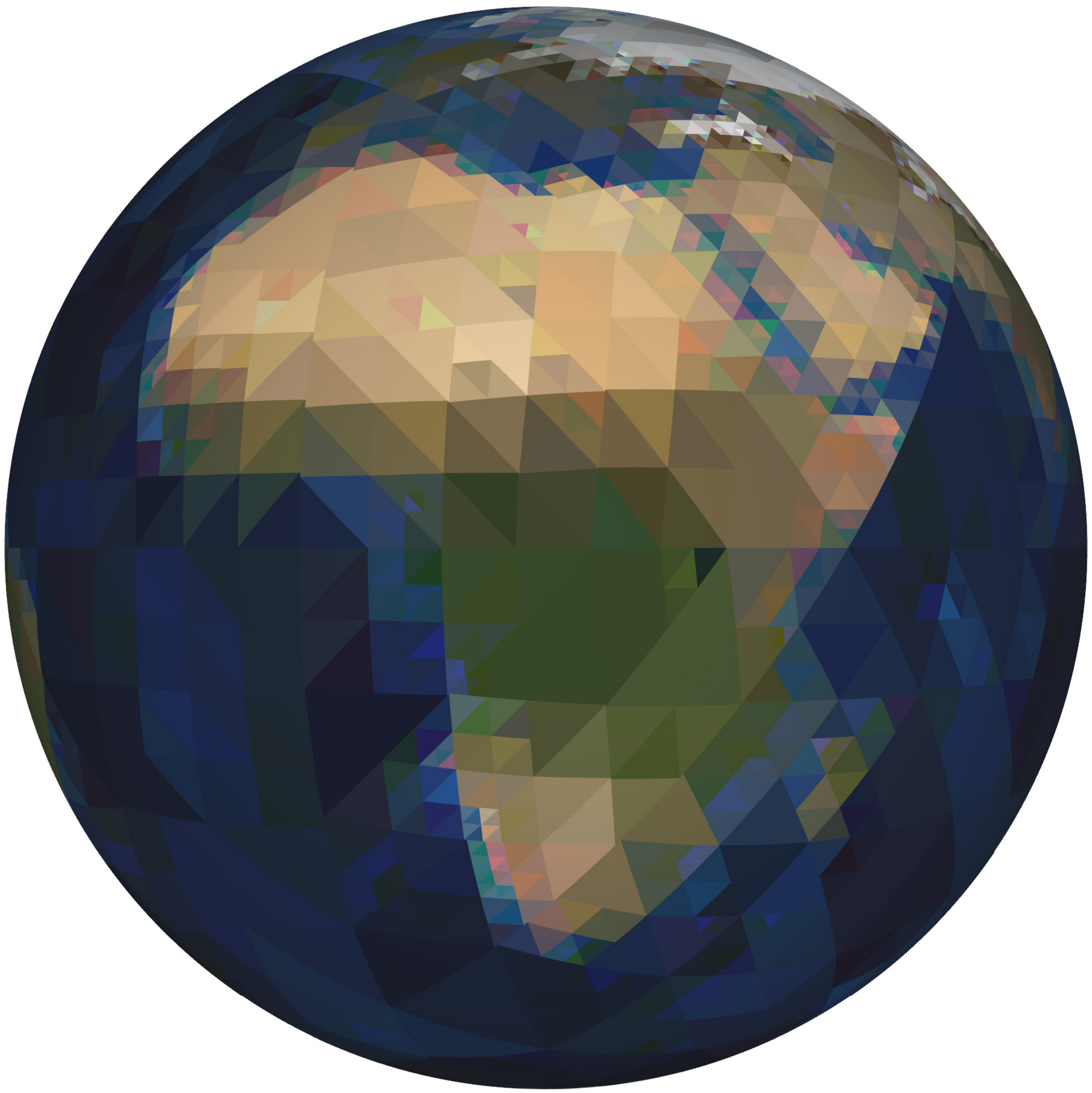


Objectives

- Efficient *representation* of spherical signals.
- Efficient *processing* of spherical signals.

⇒ Basis for $L_2(\mathbb{S}^2, d\omega)$ with:

1. Localization in space and frequency.
2. Orthogonality of basis functions.
3. Orientation invariance.

Representations for Spherical Signals

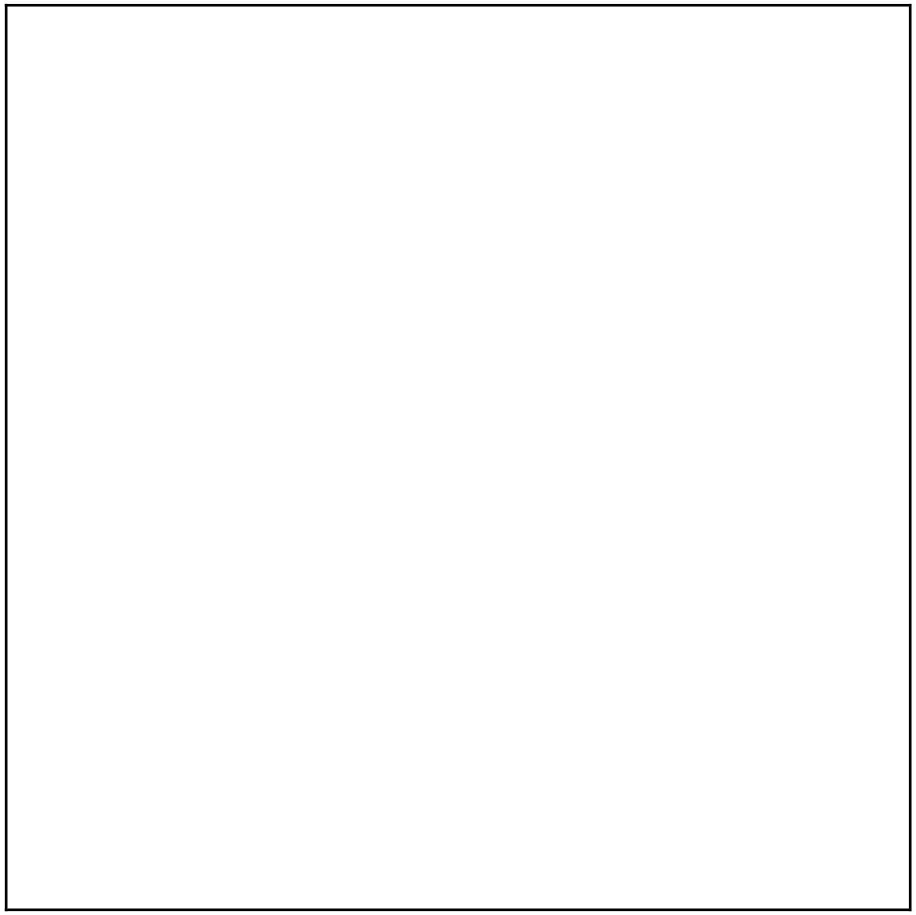
- Spherical Harmonics (SH).
- Spherical Radial Basis Functions (SRBF).
- Wavelets defined over Euclidean domains.
  [Ng et al. 2003; Ng et al. 2004; Wang et al. 2006]
- Discrete spherical wavelets.
  [Schröder and Sweldens 1995]
- Nearly orthogonal spherical Haar wavelets.
  [Nielson et al. 1997; Bonneau 1999; Rosça 2004]

Requirements

1. Localization in space and frequency.
2. Orthogonality of basis functions.
3. Orientation invariance.

$\Rightarrow$ Orthogonal and symmetric spherical Haar wavelet basis?

Haar Wavelets in 2D



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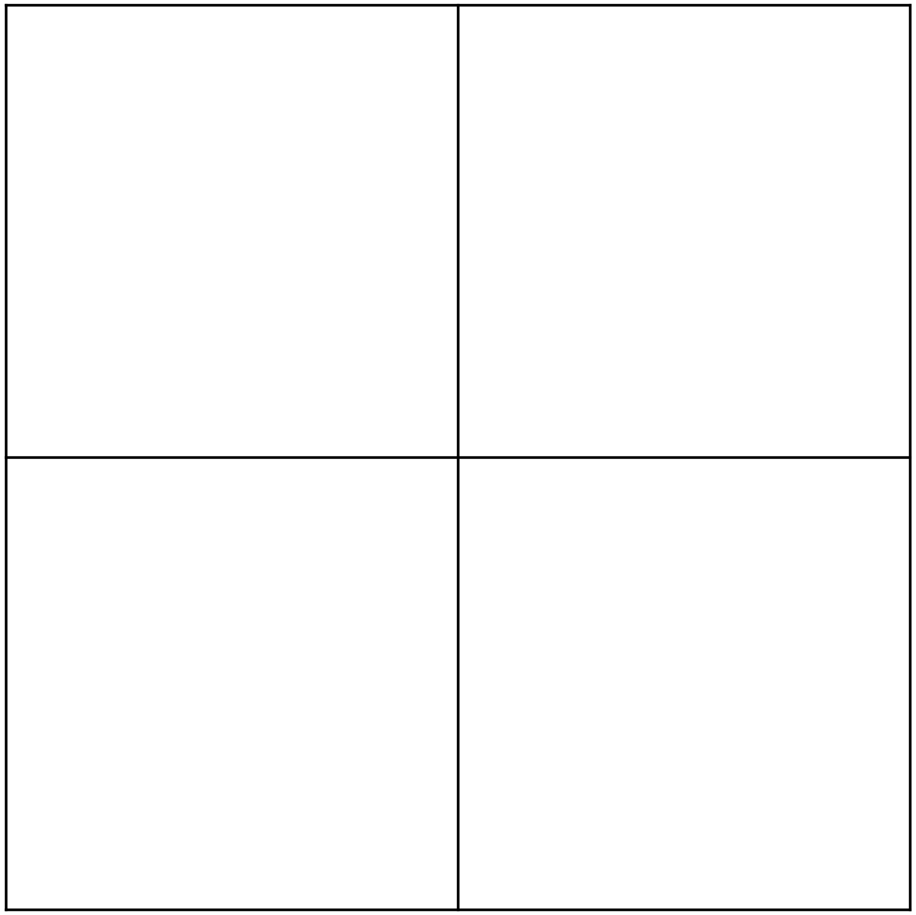

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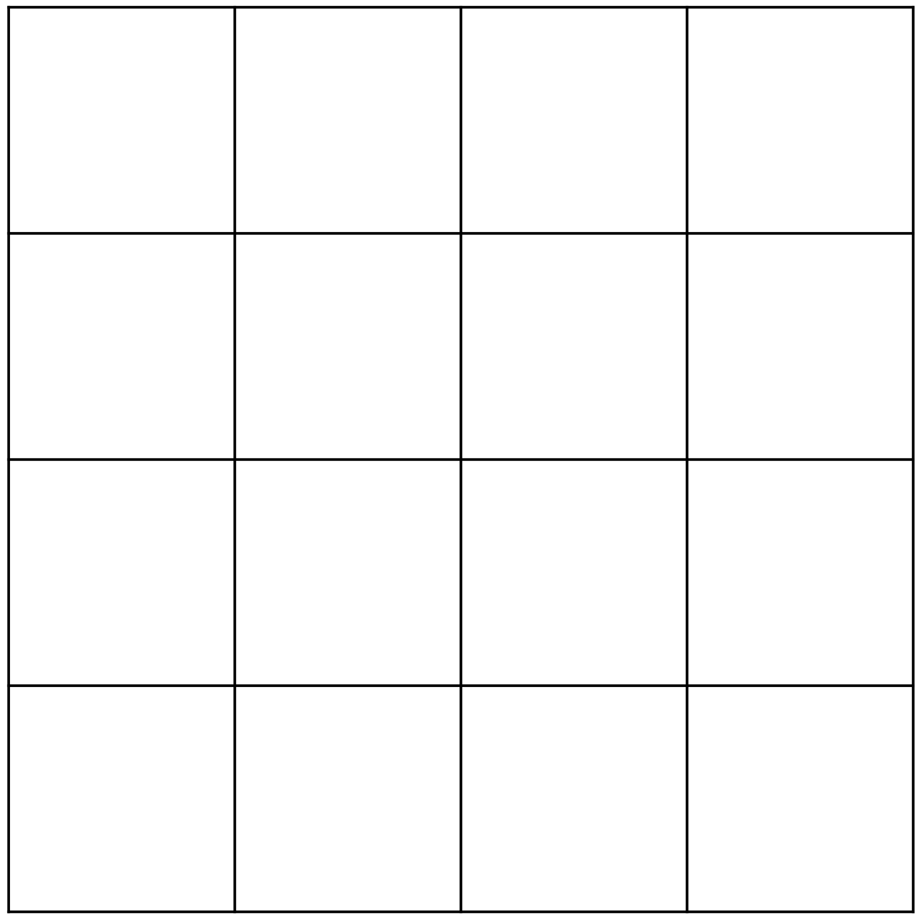

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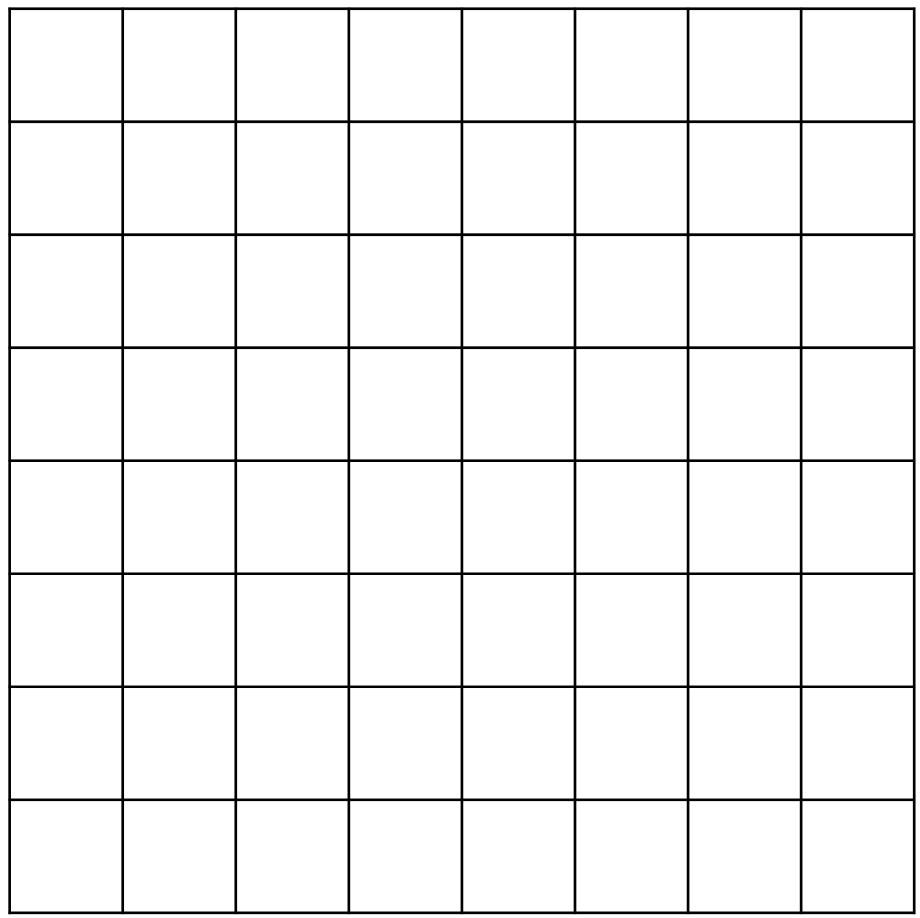

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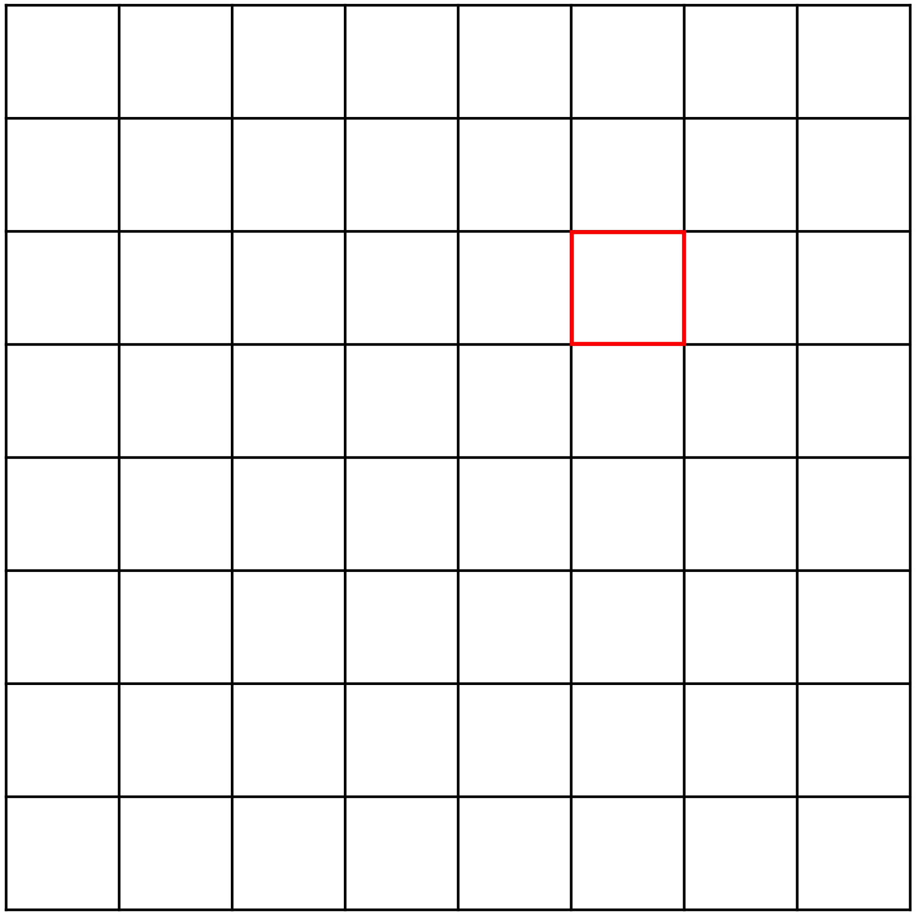

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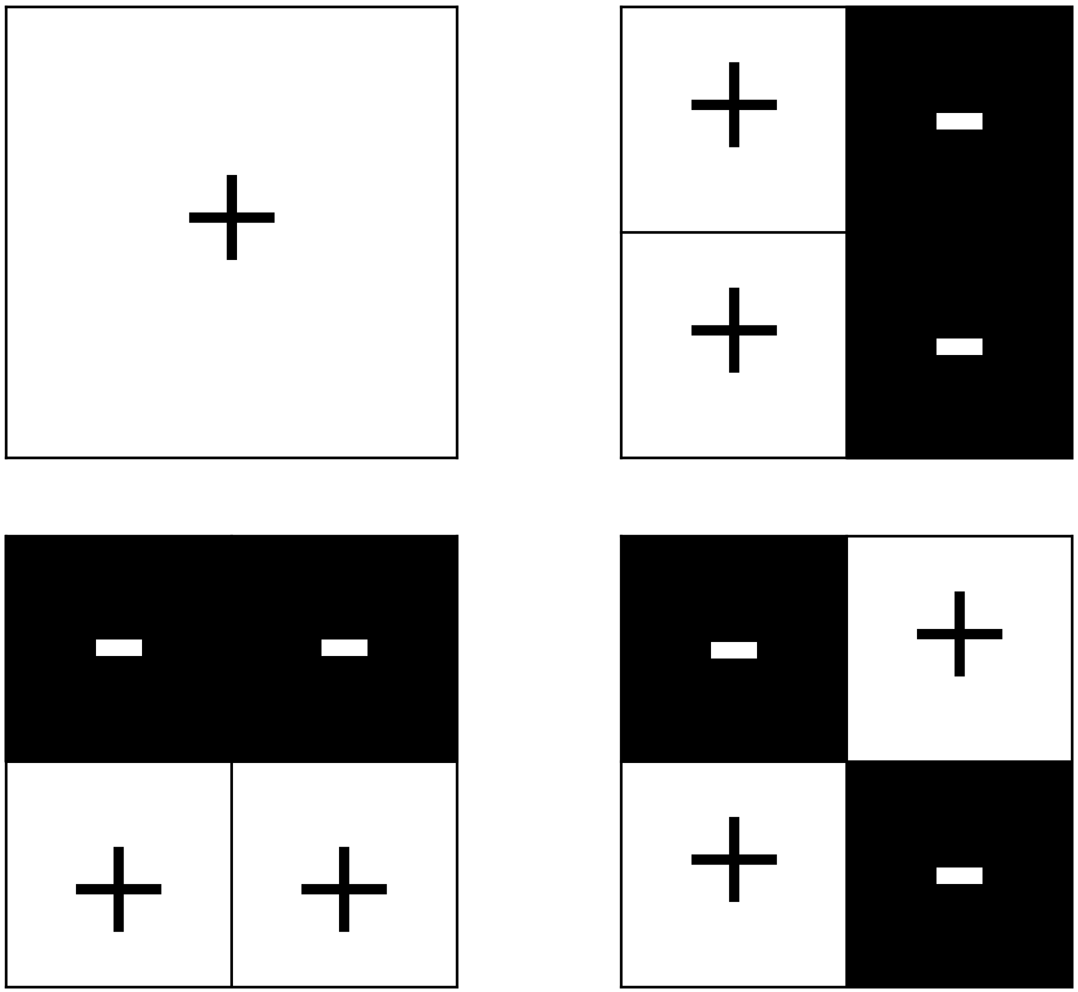

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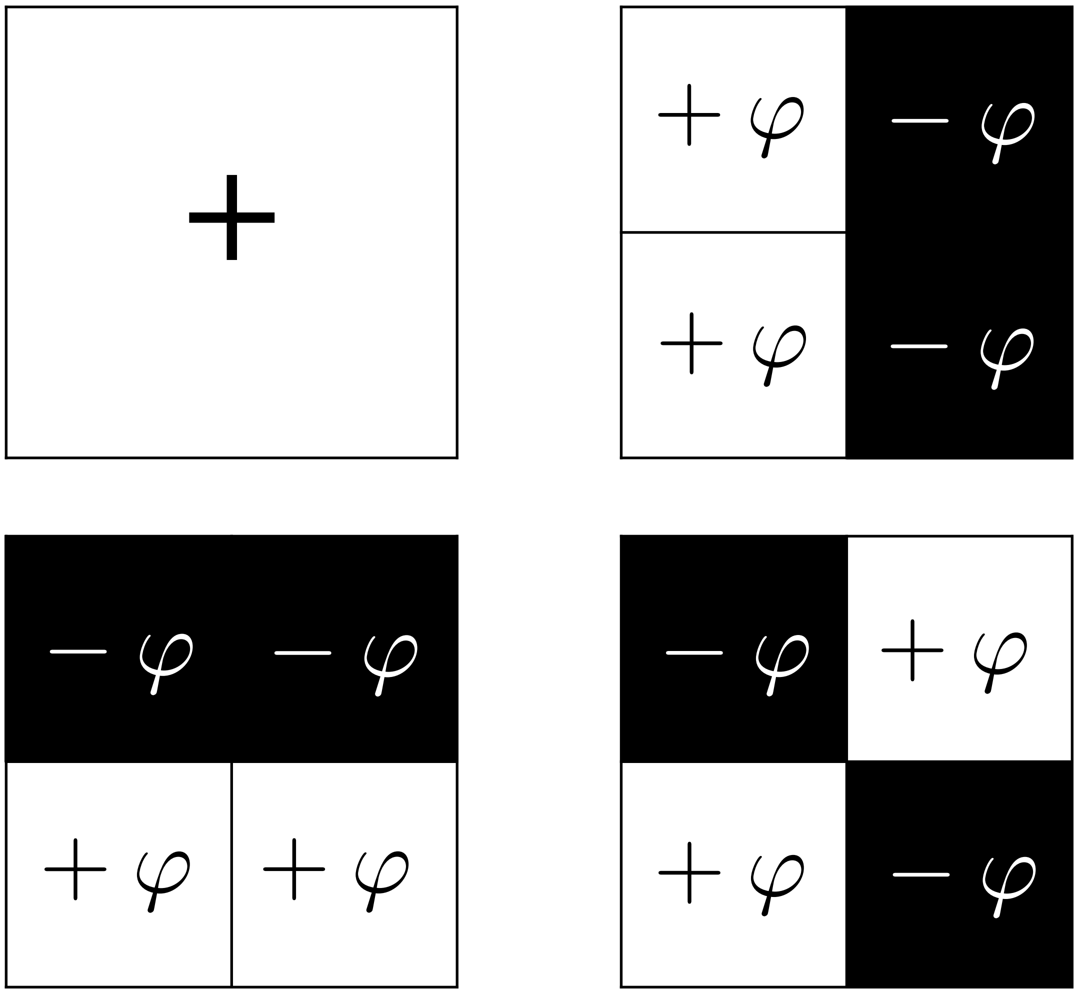

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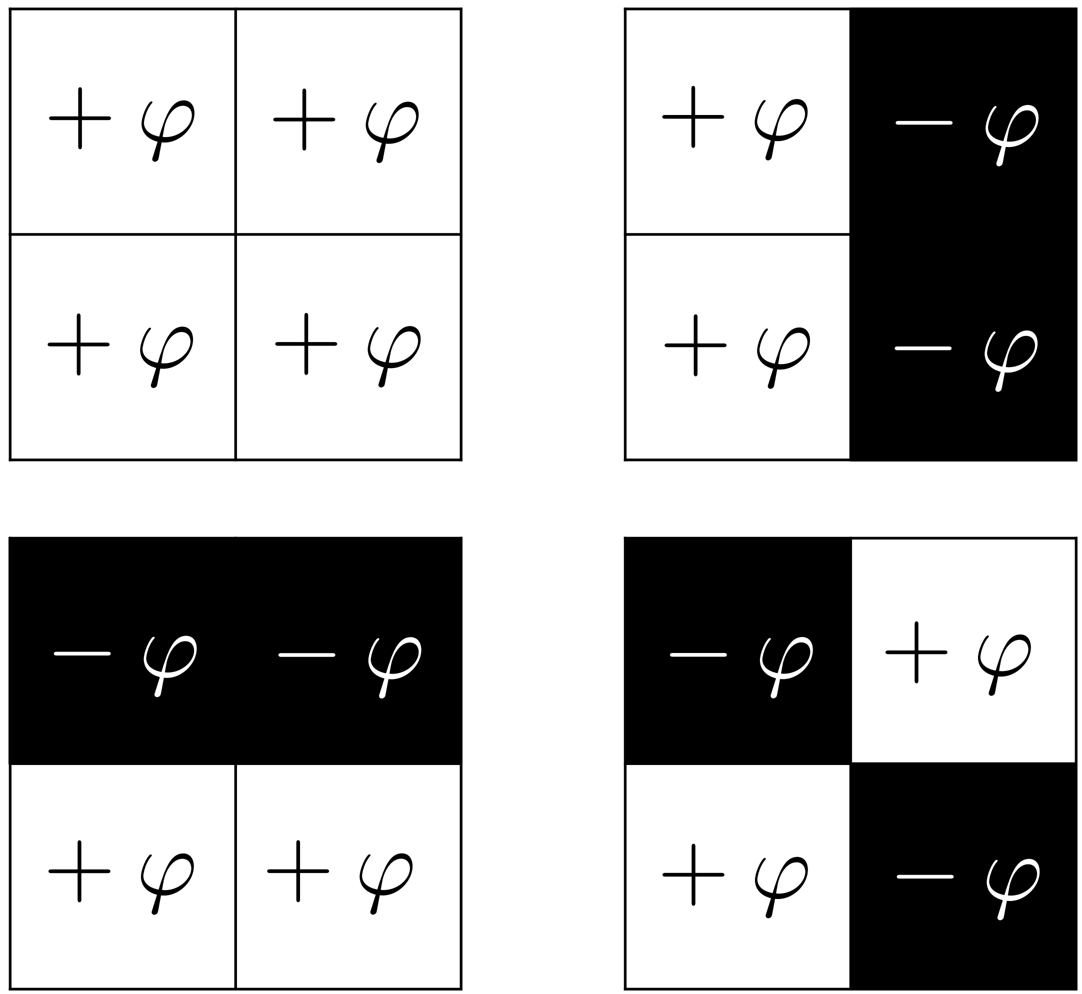
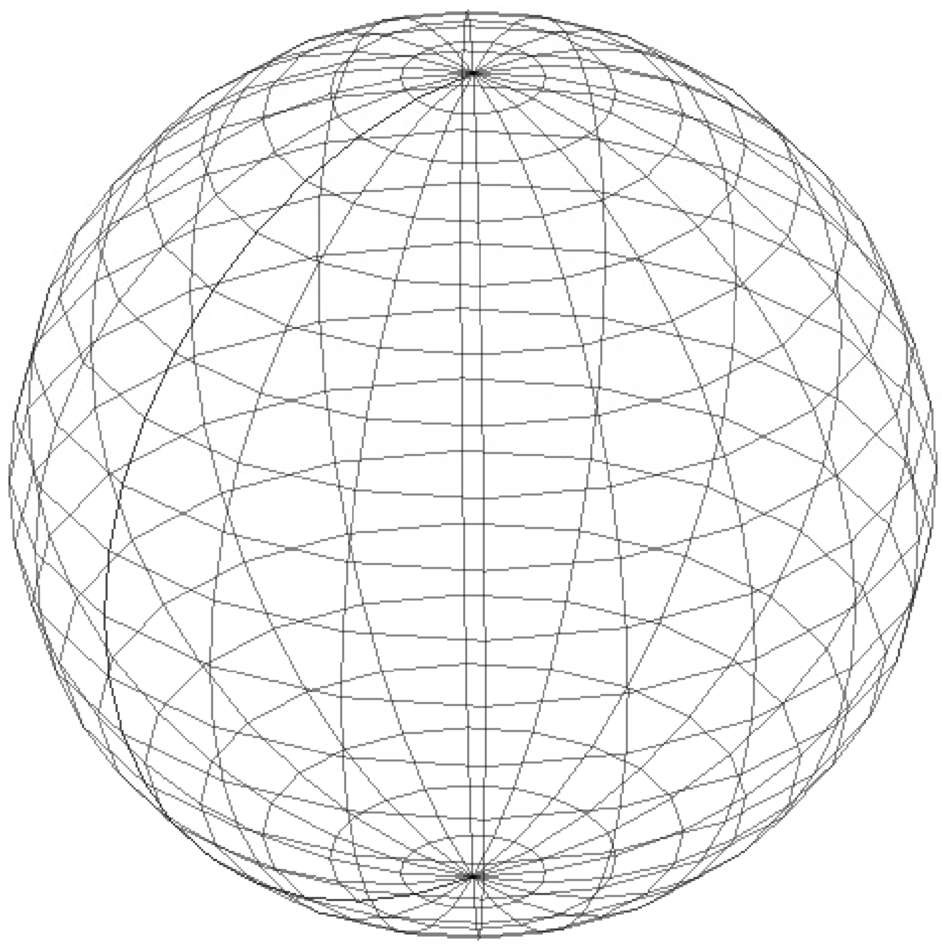


Spherical Haar Wavelets



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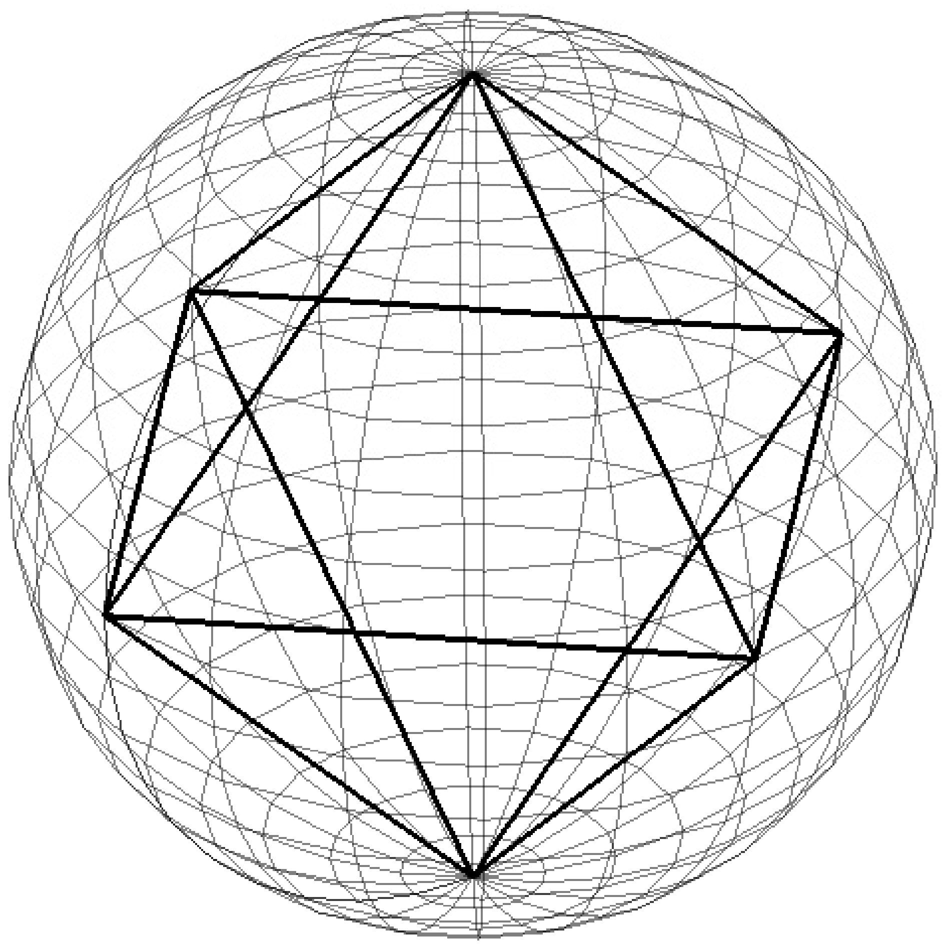

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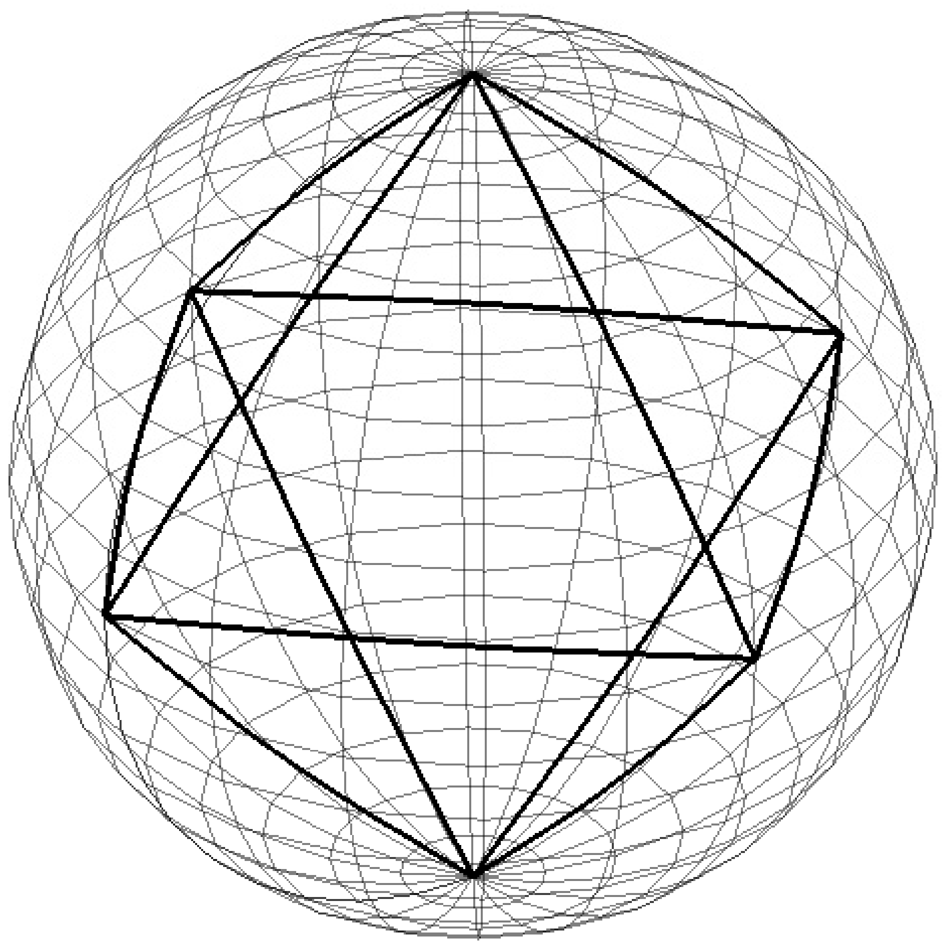

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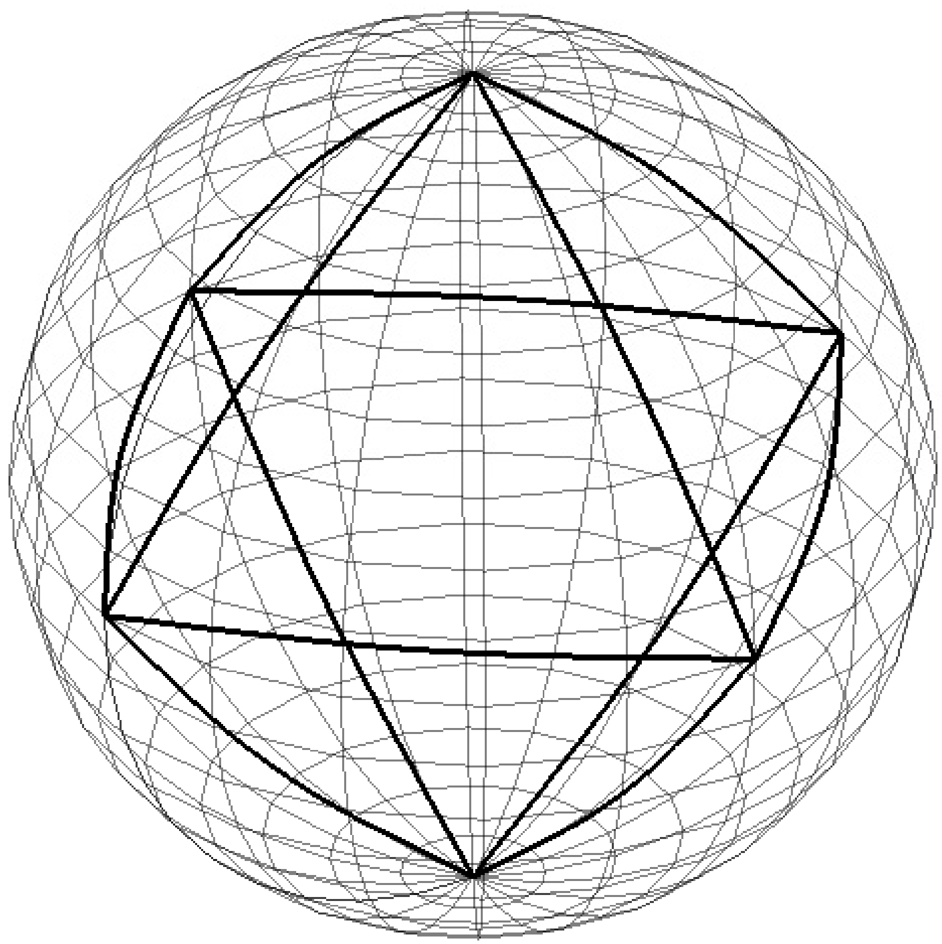

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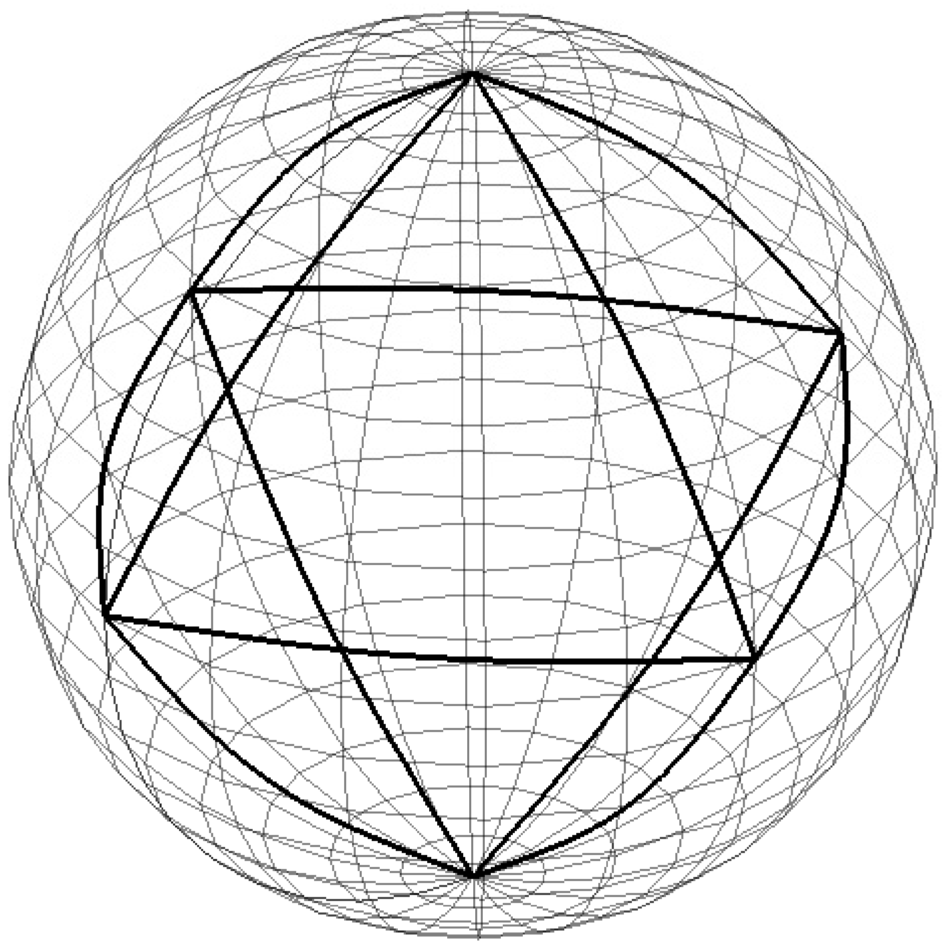

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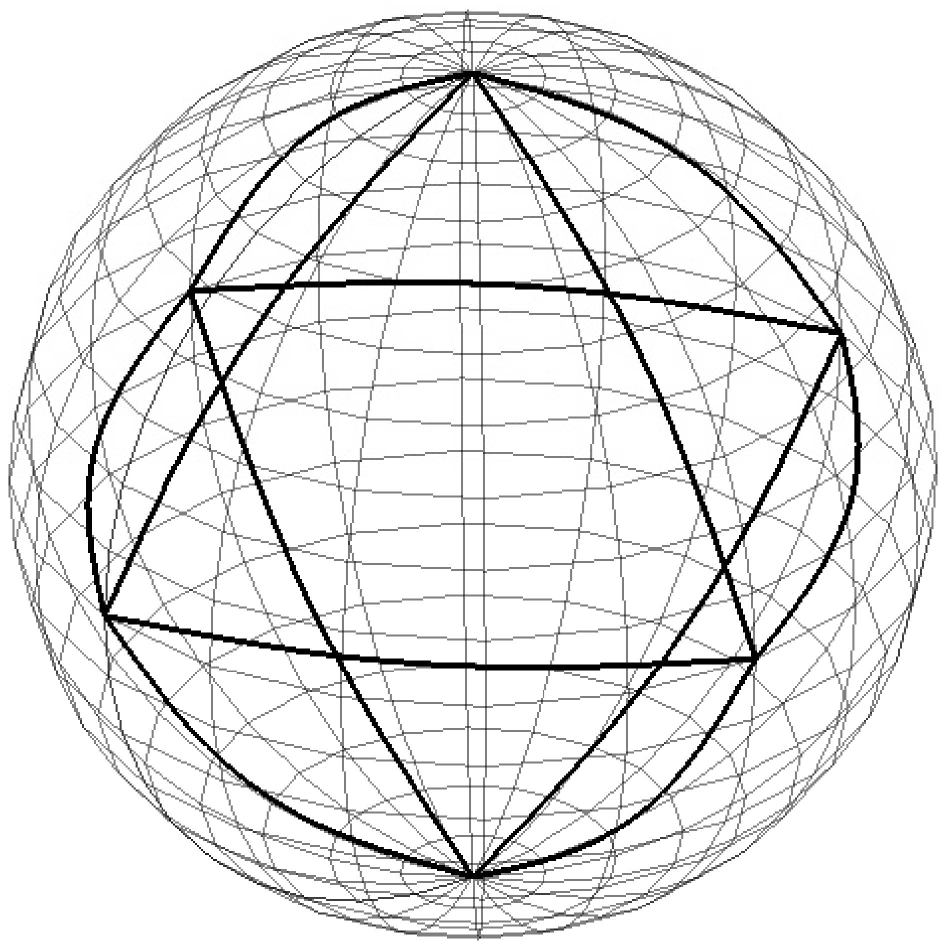

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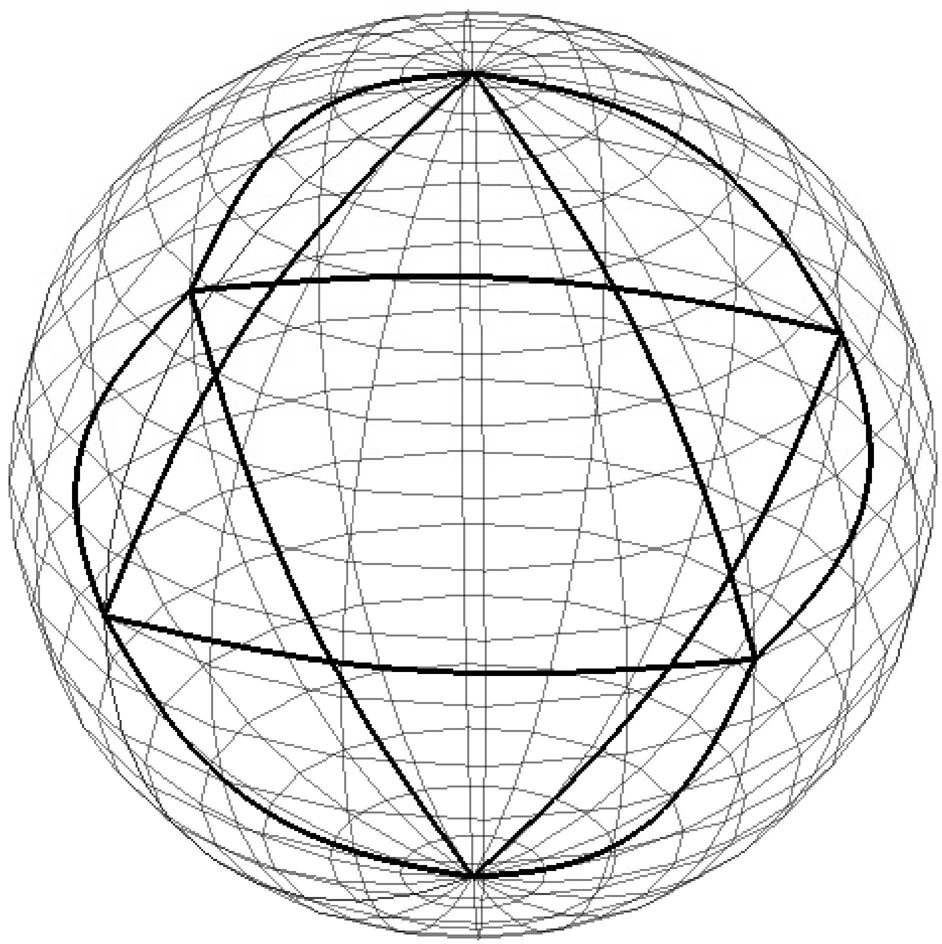

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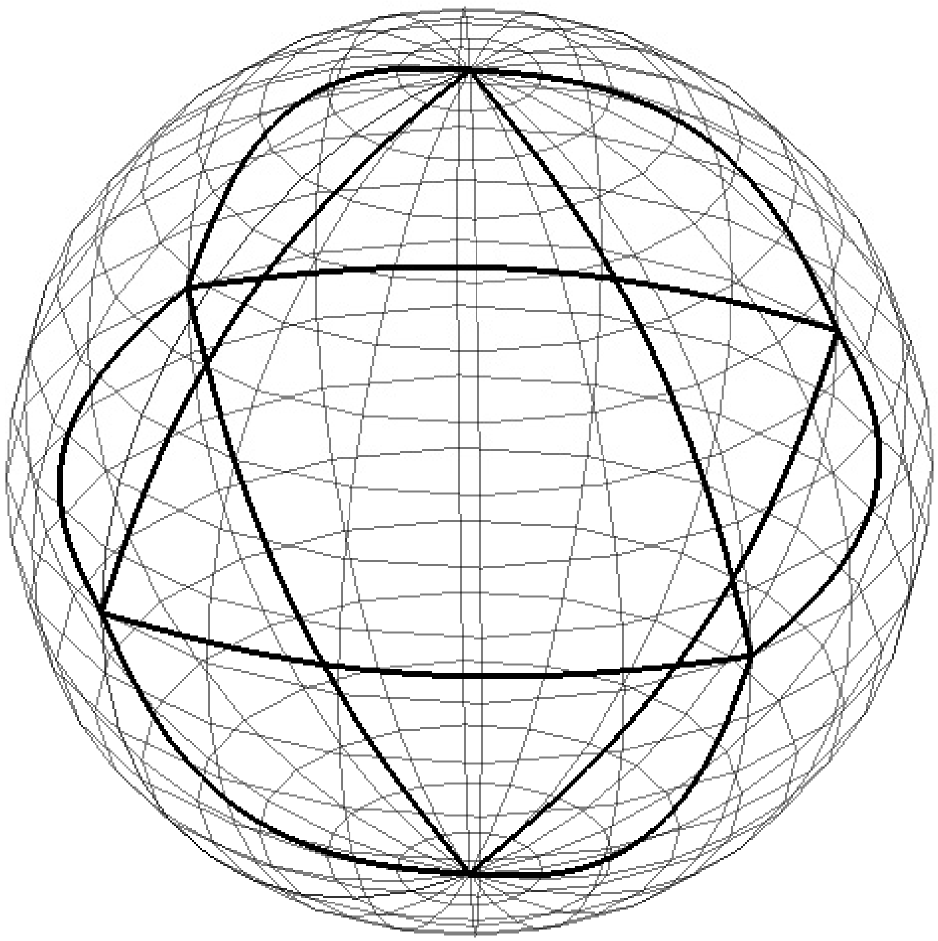

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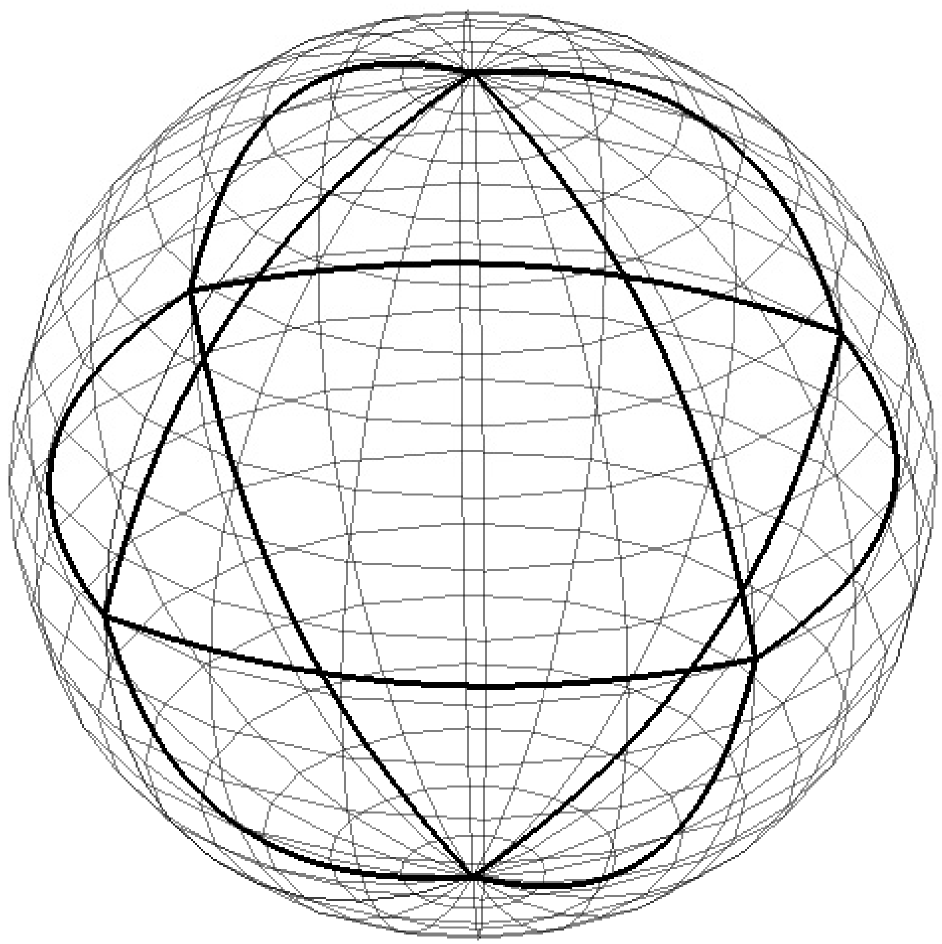

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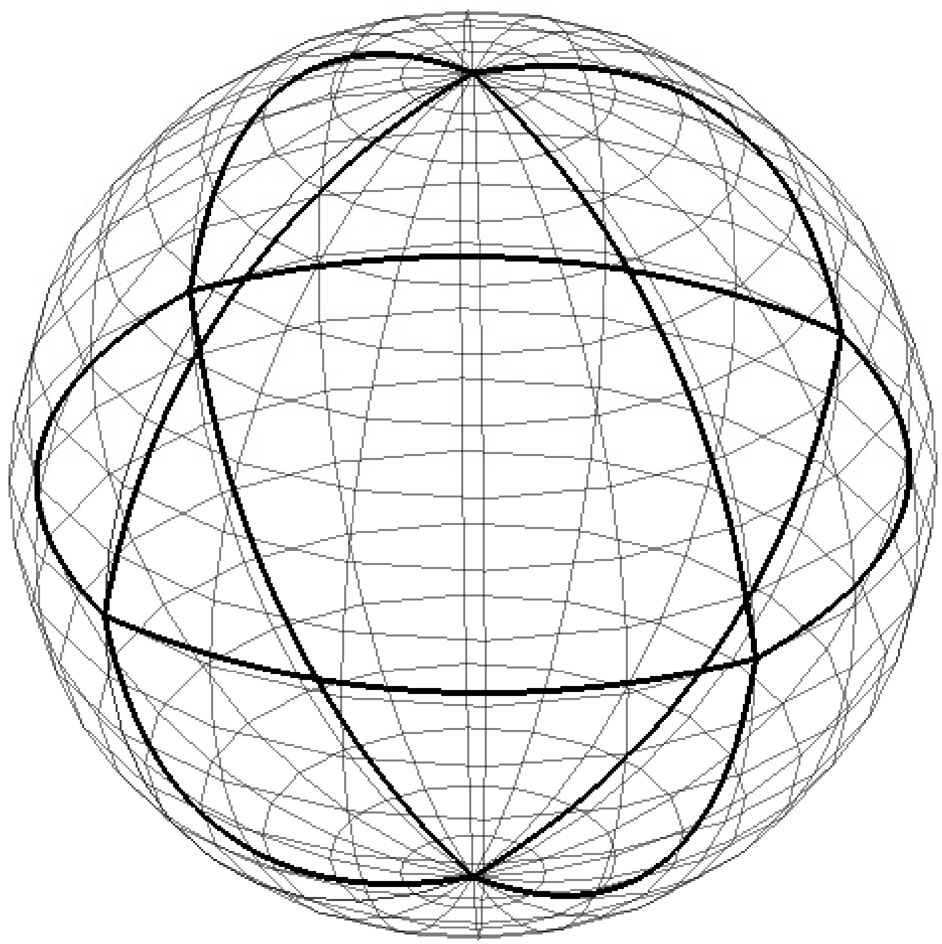

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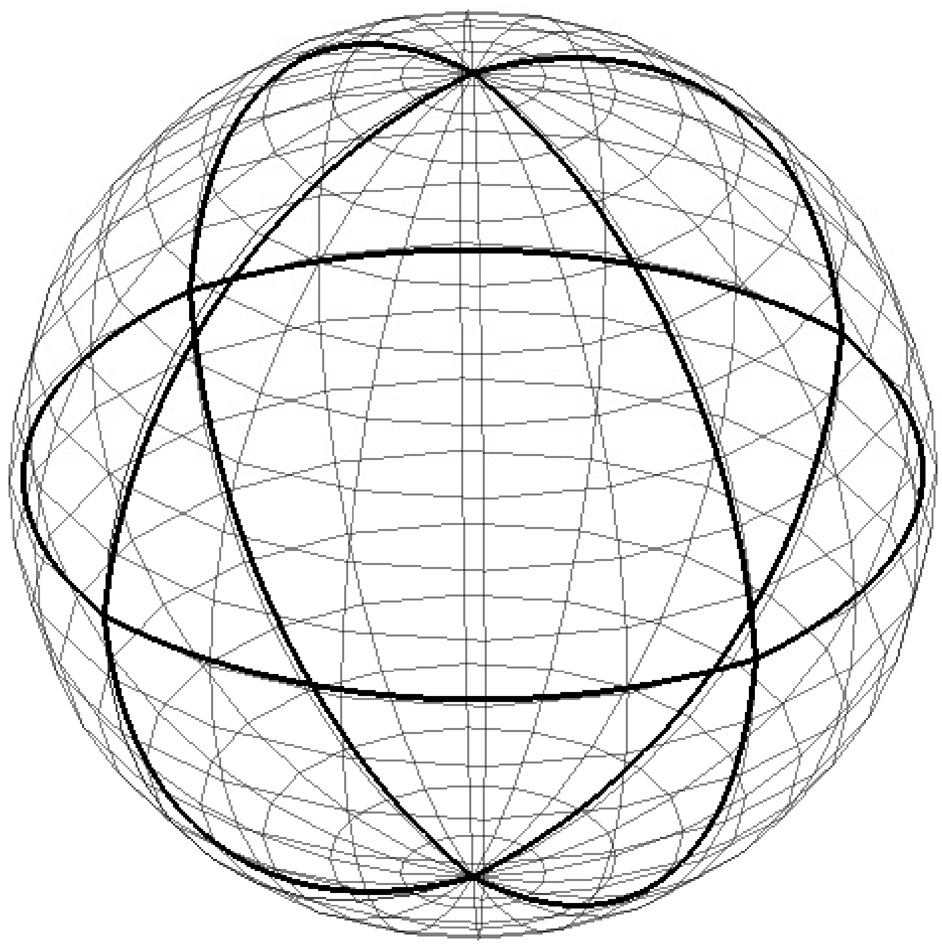

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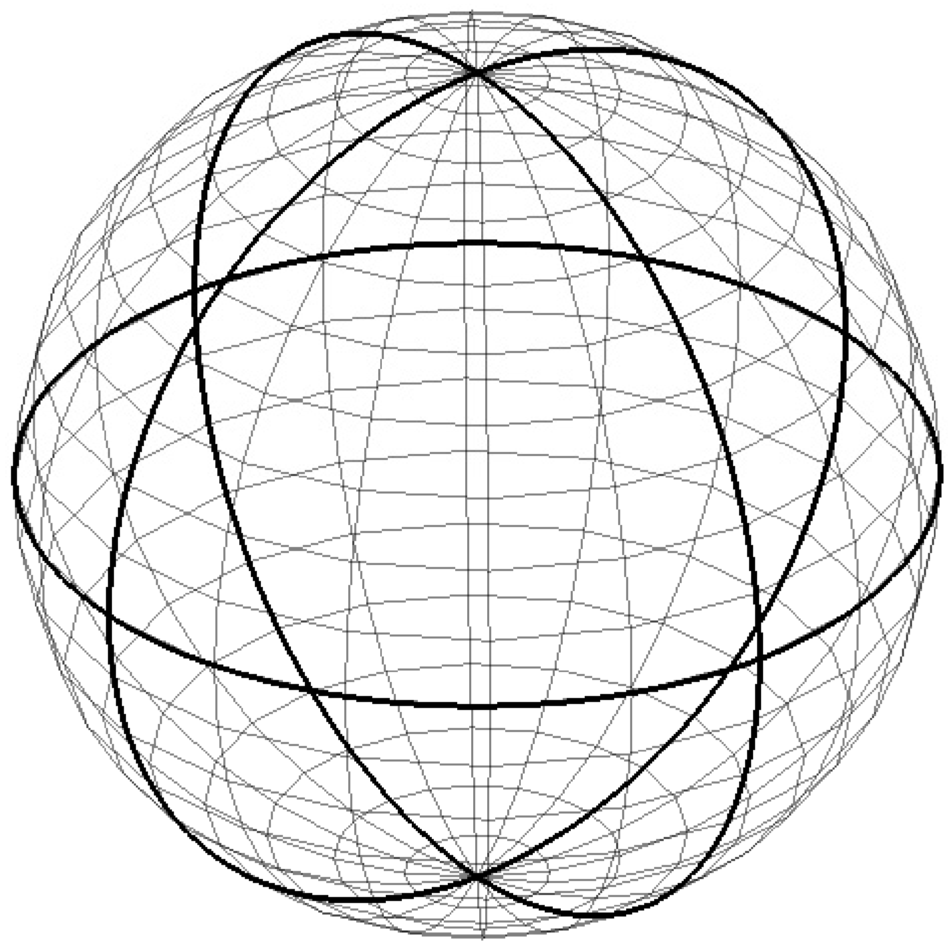

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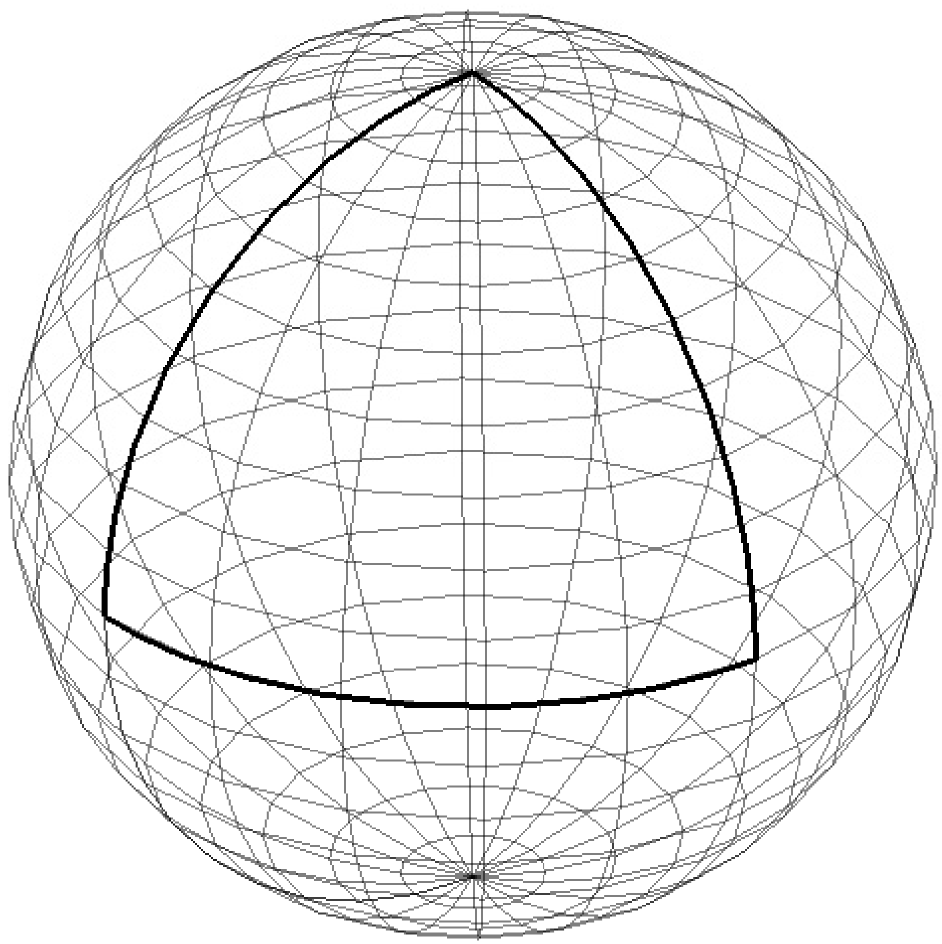

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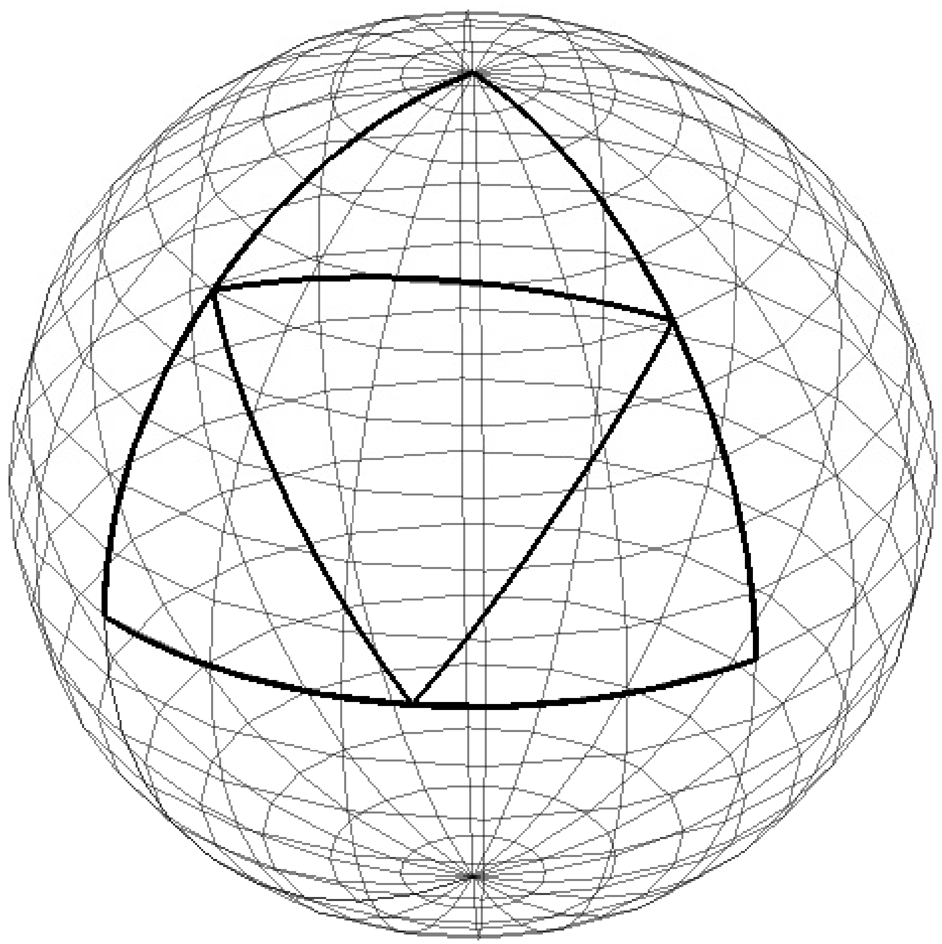

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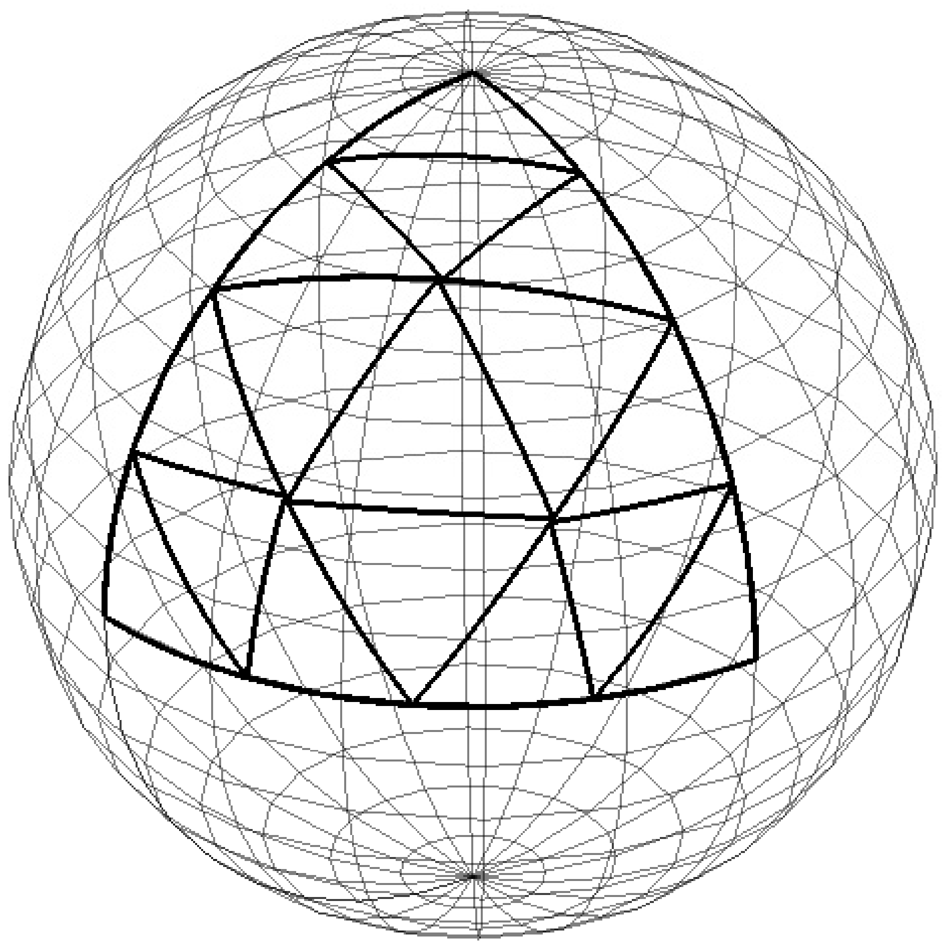

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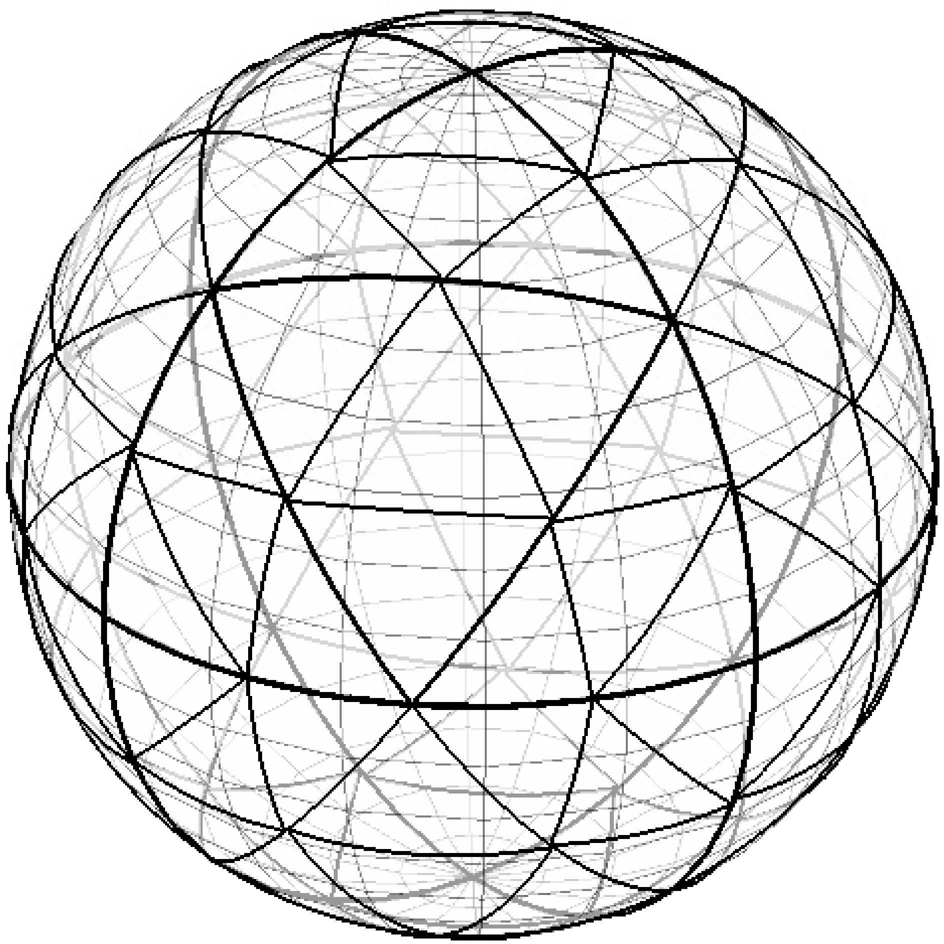

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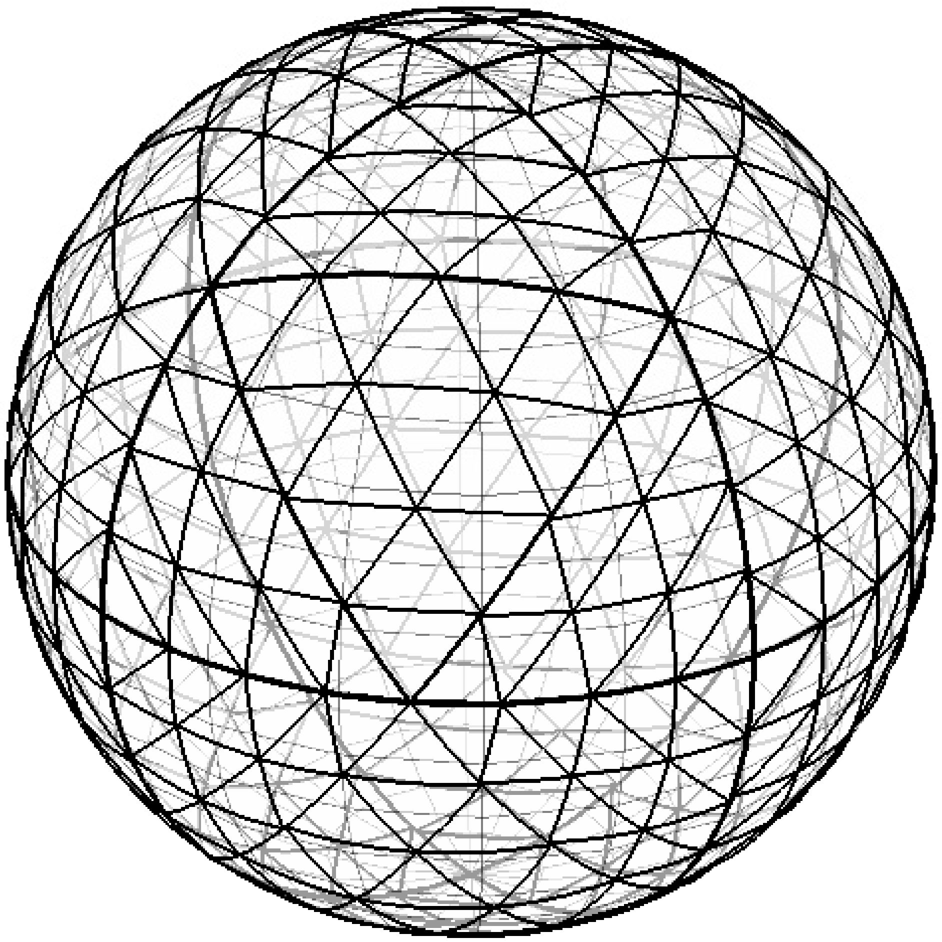

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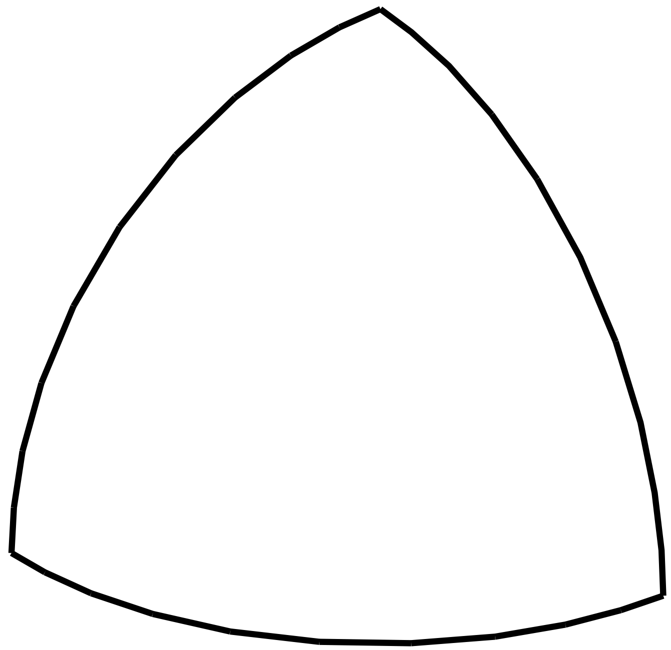

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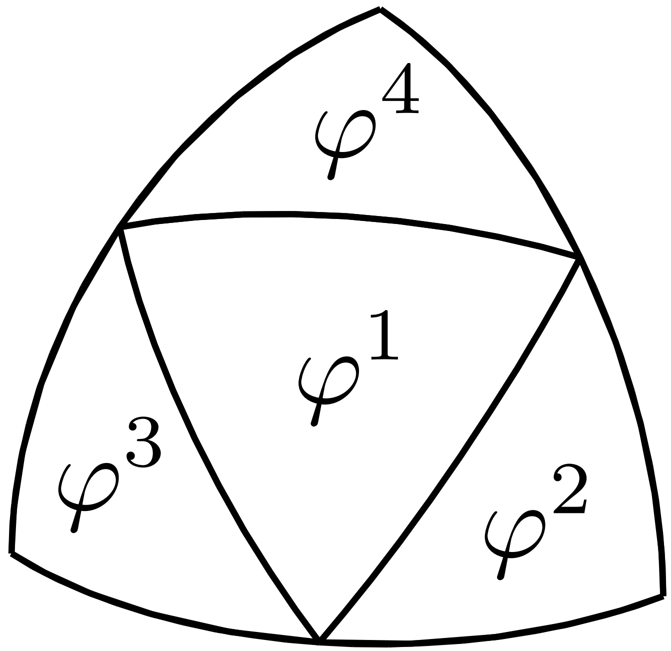

Spherical Haar Wavelets

- Scaling basis functions

$$\varphi_{j,k} = \sum_{l=1}^{4} h_{j,k,l} \, \varphi_{j,k}^{l}$$

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$$\psi_{j,k}^{i} = \sum_{l=1}^{4} g_{j,k,l}^{i}\, \varphi_{j,k}^{l}$$

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SOHO Wavelets

$$S_{j,k} = \begin{pmatrix} \frac{\sqrt{\alpha_1}}{\sqrt{\alpha_p}} & g_0^0 & g_0^1 & g_0^2 \\ \frac{\sqrt{\alpha_2}}{\sqrt{\alpha_p}} & g_1^0 & g_1^1 & g_1^2 \\ \frac{\sqrt{\alpha_3}}{\sqrt{\alpha_p}} & g_2^0 & g_2^1 & g_2^2 \\ \frac{\sqrt{\alpha_4}}{\sqrt{\alpha_p}} & g_3^0 & g_3^1 & g_3^2 \end{pmatrix}$$

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$$h_{j,k,l}$$

SOHO Wavelets

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$$g^i_{j,k,l}$$

SOHO Wavelets

- Semi-orthogonal wavelet basis:

$$\left\langle \psi^0_{j,k} \, , \, \varphi_{j,k} \right\rangle = \left\langle \psi^1_{j,k} \, , \, \varphi_{j,k} \right\rangle = \left\langle \psi^2_{j,k} \, , \, \varphi_{j,k} \right\rangle = 0$$

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⇒ Matrix notation:

$$[\langle \Phi_{j,k} \mid \Psi_{j,k} \rangle] = 0$$

$$[\langle \Phi_{j,k} \mid \Phi_{j+1,k} \rangle] \, G_{j,k} = 0$$

SOHO Wavelets

- Semi-orthogonal spherical Haar wavelets:

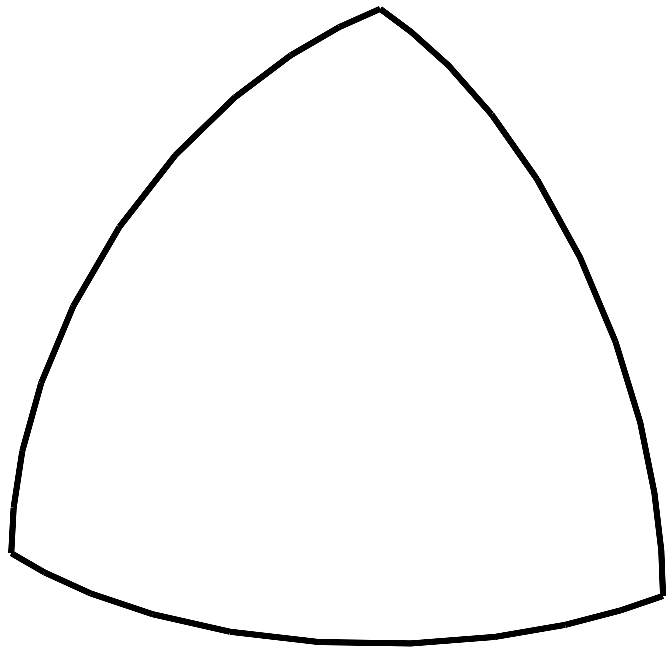
$$\hat{S}_{j,k} = \begin{pmatrix} \frac{\sqrt{\alpha_1}}{\sqrt{\alpha_p}} & \frac{\sqrt{\alpha_2}}{\sqrt{\alpha_1}} & \frac{\sqrt{\alpha_3}}{\sqrt{\alpha_2}} & \frac{\sqrt{\alpha_4}}{\sqrt{\alpha_3}} \\ \frac{\sqrt{\alpha_2}}{\sqrt{\alpha_p}} & 1 & 0 & 0 \\ \frac{\sqrt{\alpha_3}}{\sqrt{\alpha_p}} & 0 & 1 & 0 \\ \frac{\sqrt{\alpha_4}}{\sqrt{\alpha_p}} & 0 & 0 & 1 \end{pmatrix}$$

SOHO Wavelets

- Symmetric, orthogonal spherical Haar wavelets?

$$\hat{S}_{j,k} = \begin{pmatrix} \frac{\sqrt{\alpha_1}}{\sqrt{\alpha_p}} & -a_{1,2}\,\frac{\sqrt{\alpha_2}}{\sqrt{\alpha_1}} & -a_{1,3}\,\frac{\sqrt{\alpha_3}}{\sqrt{\alpha_2}} & -a_{1,4}\,\frac{\sqrt{\alpha_4}}{\sqrt{\alpha_3}} \\ \frac{\sqrt{\alpha_2}}{\sqrt{\alpha_p}} & a_{2,2} & a_{2,3} & a_{2,4} \\ \frac{\sqrt{\alpha_3}}{\sqrt{\alpha_p}} & a_{3,2} & a_{3,3} & a_{3,4} \\ \frac{\sqrt{\alpha_4}}{\sqrt{\alpha_p}} & a_{4,2} & a_{4,3} & a_{4,4} \end{pmatrix}$$

SOHO Wavelets



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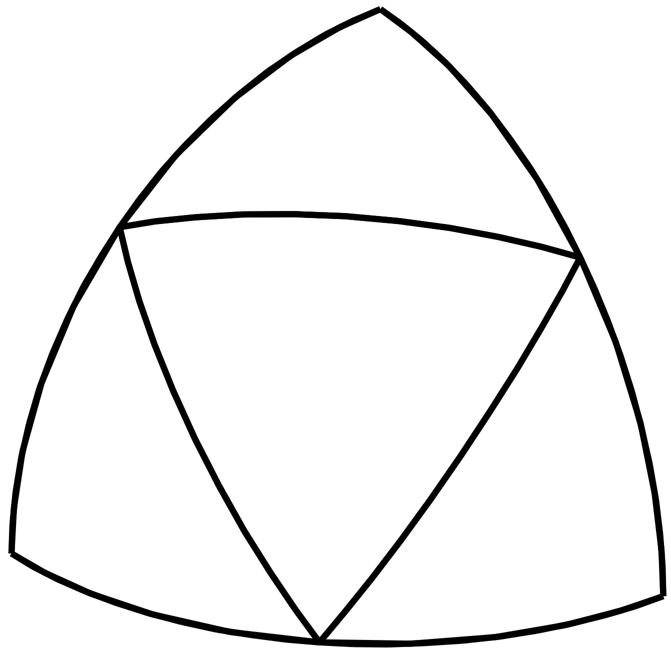

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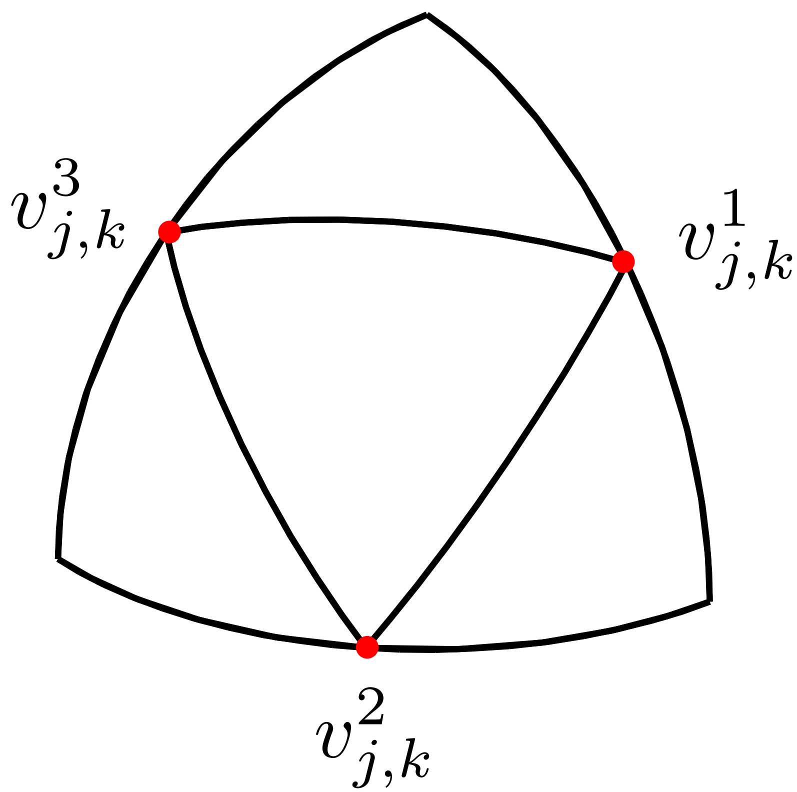

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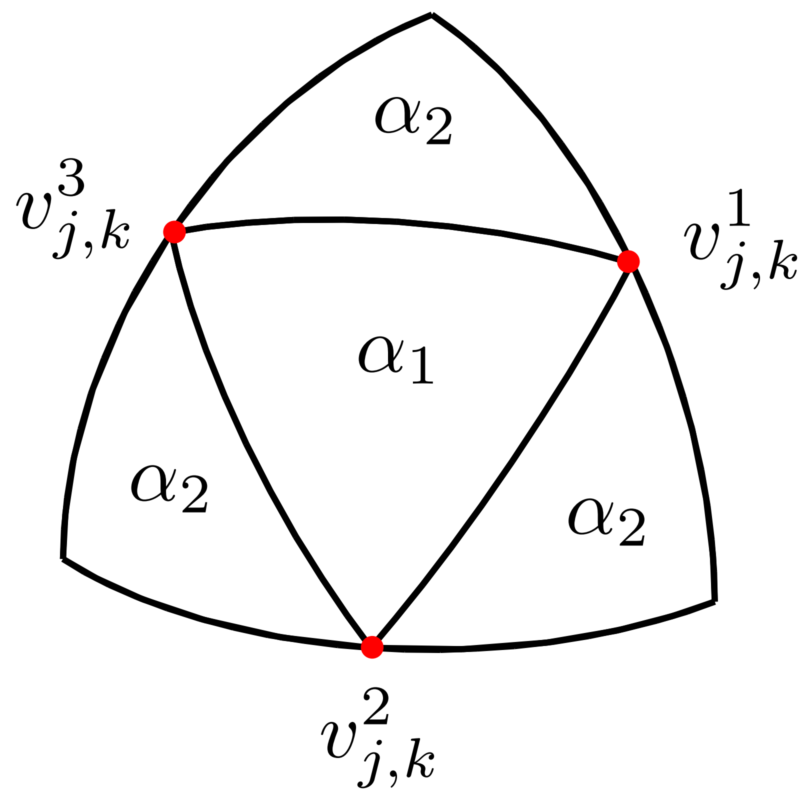


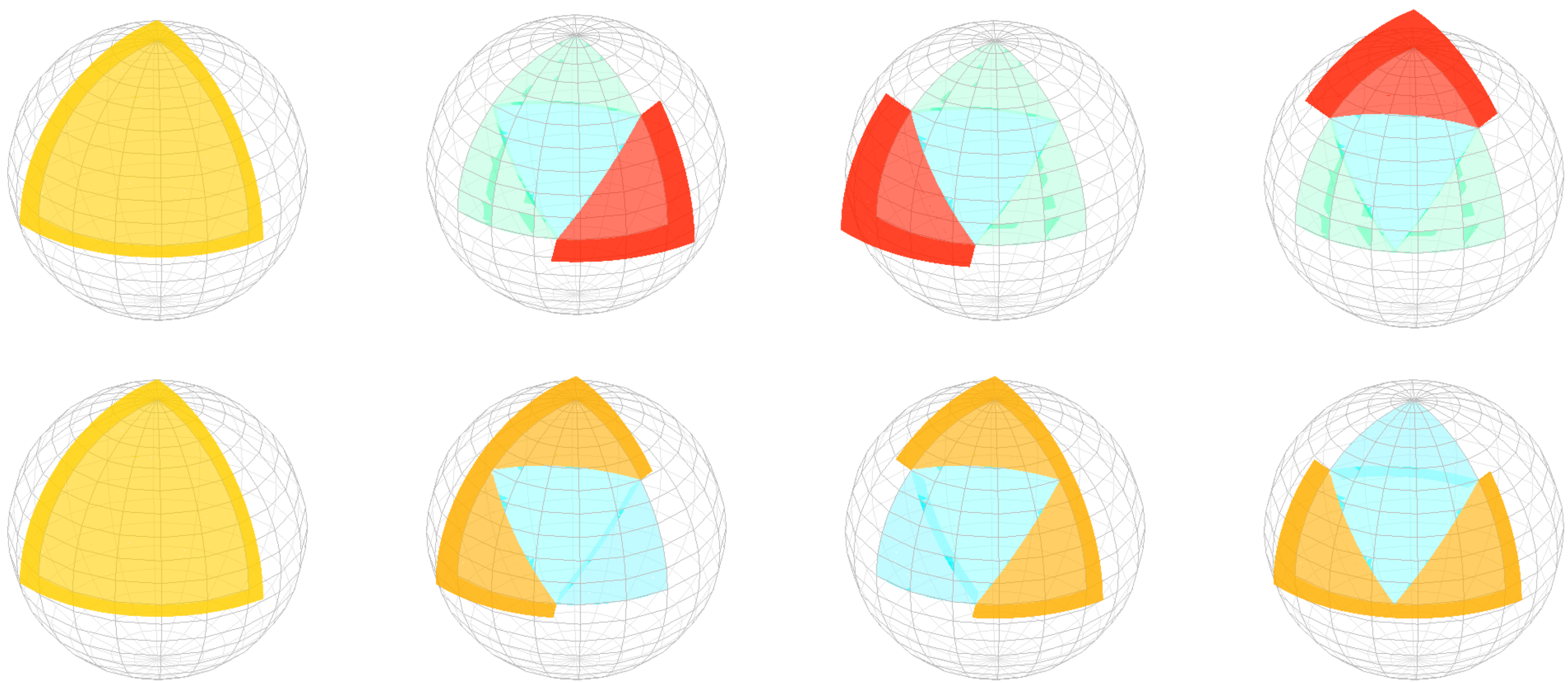
SOHO Wavelets

$$\hat{S}_{j,k} = \begin{pmatrix} \frac{\sqrt{\alpha_1}}{\sqrt{\alpha_p}} & -a_{1,2}\,\frac{\sqrt{\alpha_2}}{\sqrt{\alpha_1}} & -a_{1,3}\,\frac{\sqrt{\alpha_3}}{\sqrt{\alpha_2}} & -a_{1,4}\,\frac{\sqrt{\alpha_4}}{\sqrt{\alpha_3}} \\ \frac{\sqrt{\alpha_2}}{\sqrt{\alpha_p}} & a_{2,2} & a_{2,3} & a_{2,4} \\ \frac{\sqrt{\alpha_3}}{\sqrt{\alpha_p}} & a_{3,2} & a_{3,3} & a_{3,4} \\ \frac{\sqrt{\alpha_4}}{\sqrt{\alpha_p}} & a_{4,2} & a_{4,3} & a_{4,4} \end{pmatrix}$$

SOHO Wavelets

$$\hat{S}_{j,k} = \begin{pmatrix} \frac{\sqrt{\alpha_0}}{\sqrt{\alpha_p}} & -c\frac{\sqrt{\alpha_1}}{\sqrt{\alpha_0}} & -c\frac{\sqrt{\alpha_1}}{\sqrt{\alpha_0}} & -c\frac{\sqrt{\alpha_1}}{\sqrt{\alpha_0}} \\ \frac{\sqrt{\alpha_1}}{\sqrt{\alpha_p}} & b & a & a \\ \frac{\sqrt{\alpha_1}}{\sqrt{\alpha_p}} & a & b & a \\ \frac{\sqrt{\alpha_1}}{\sqrt{\alpha_p}} & a & a & b \end{pmatrix}$$

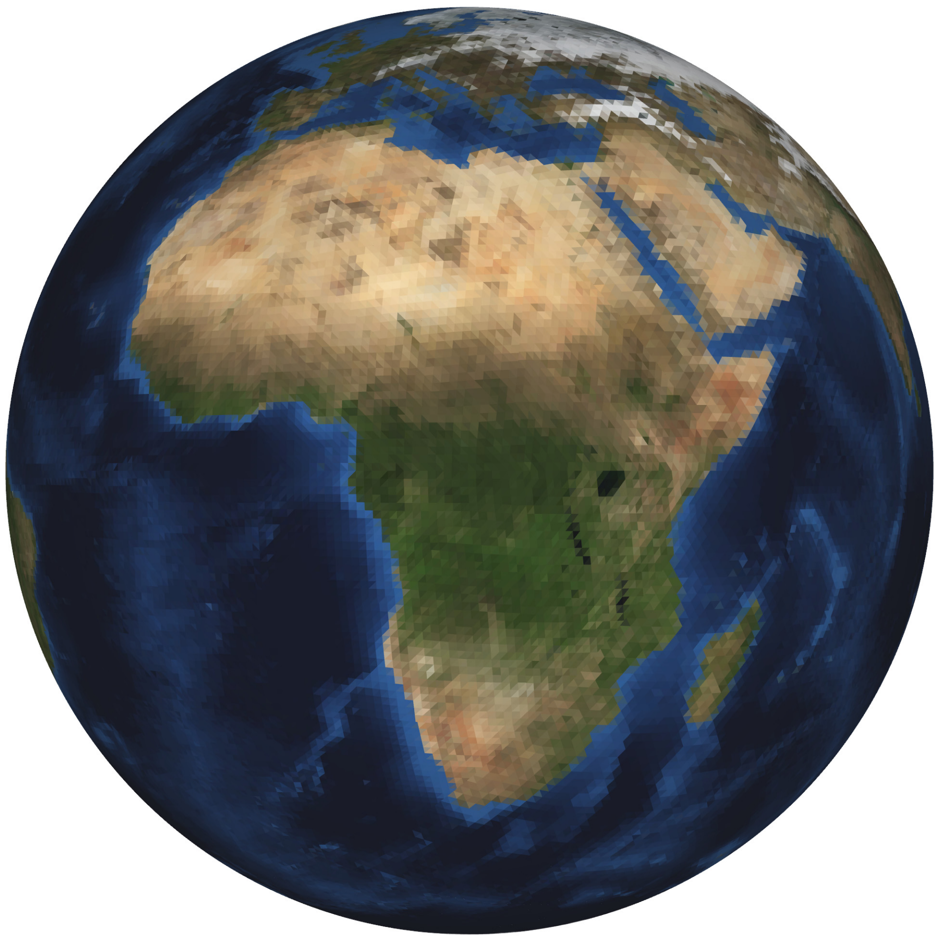
SOHO Wavelets



Experiments

- Comparison of
  - SOHO wavelets,
  - Bio-Haar wavelets [Schröder and Sweldens1995],
  - Pseudo Haar wavelets [Ma et al. 2006],
  - Four nearly orthogonal spherical Haar wavelet bases [Nielson et al. 1997; Bonneau 1999].

Experiments



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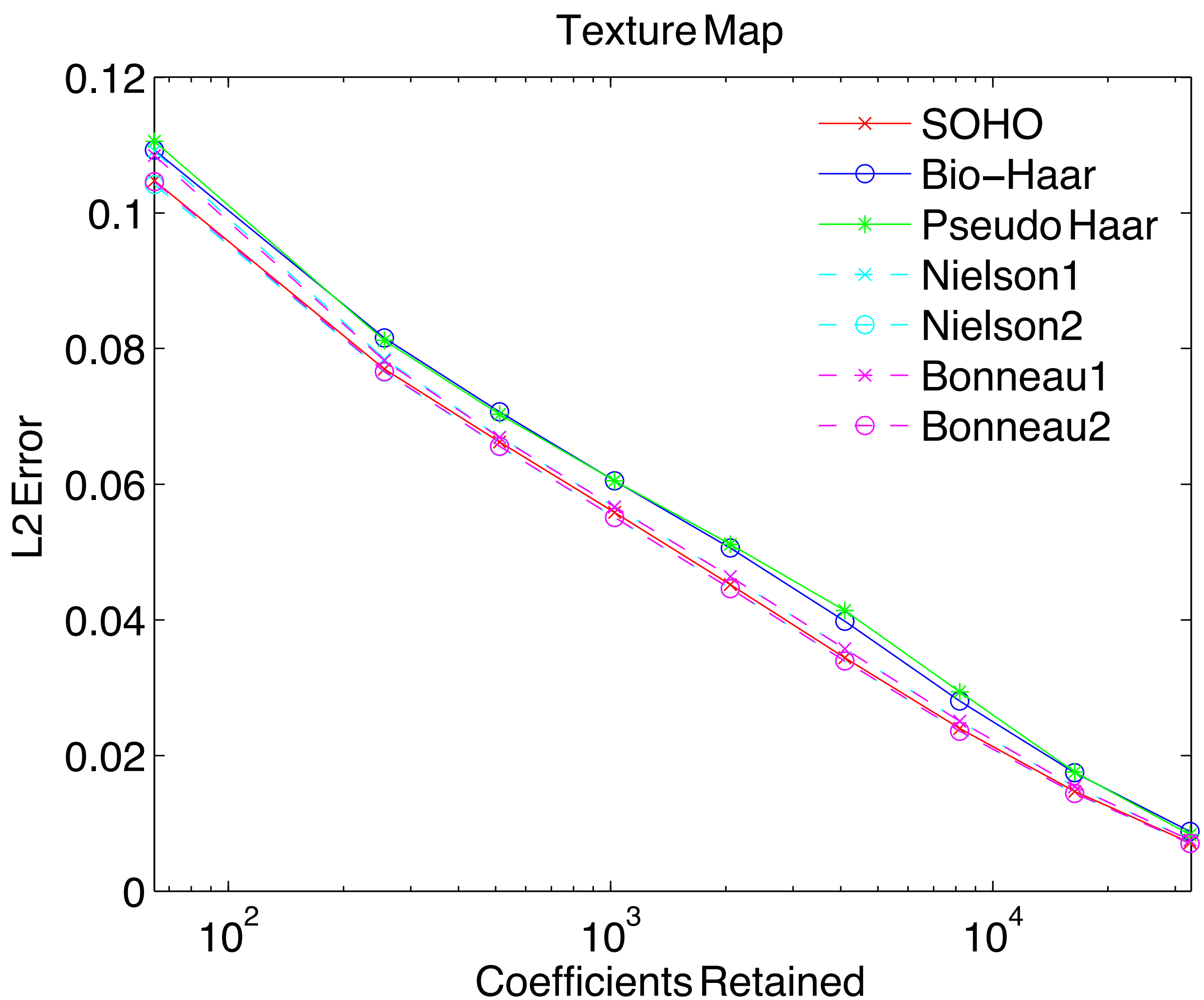

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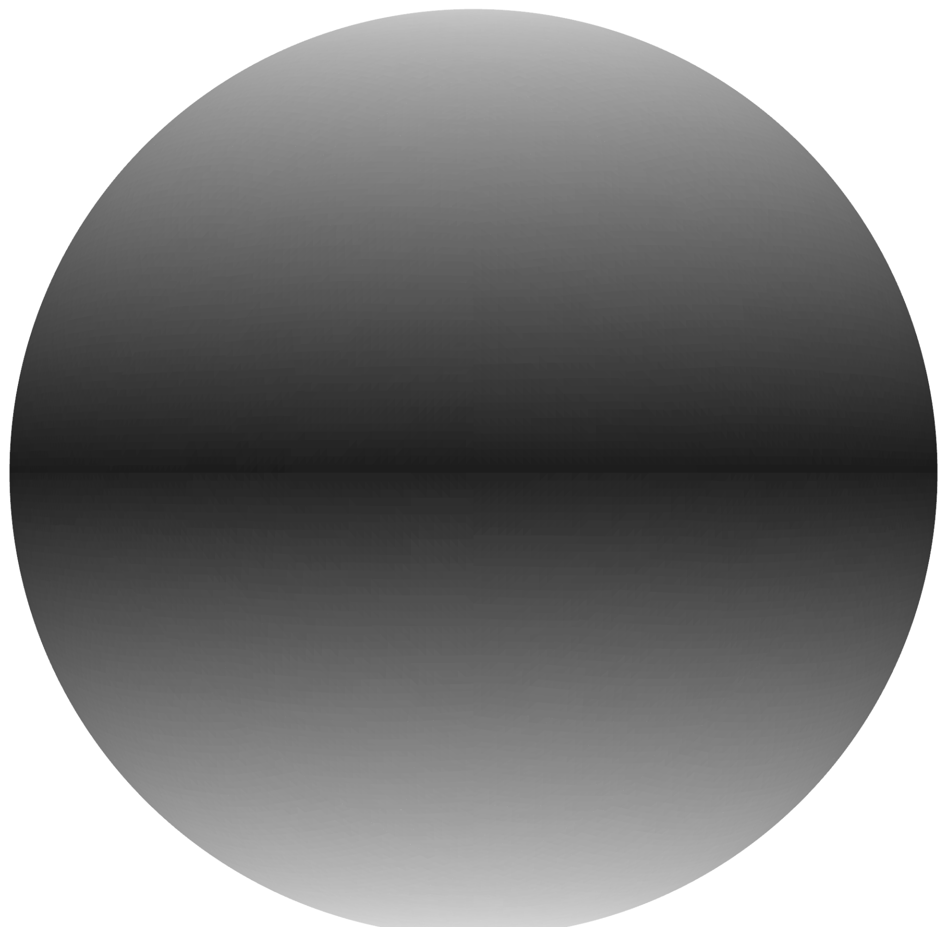

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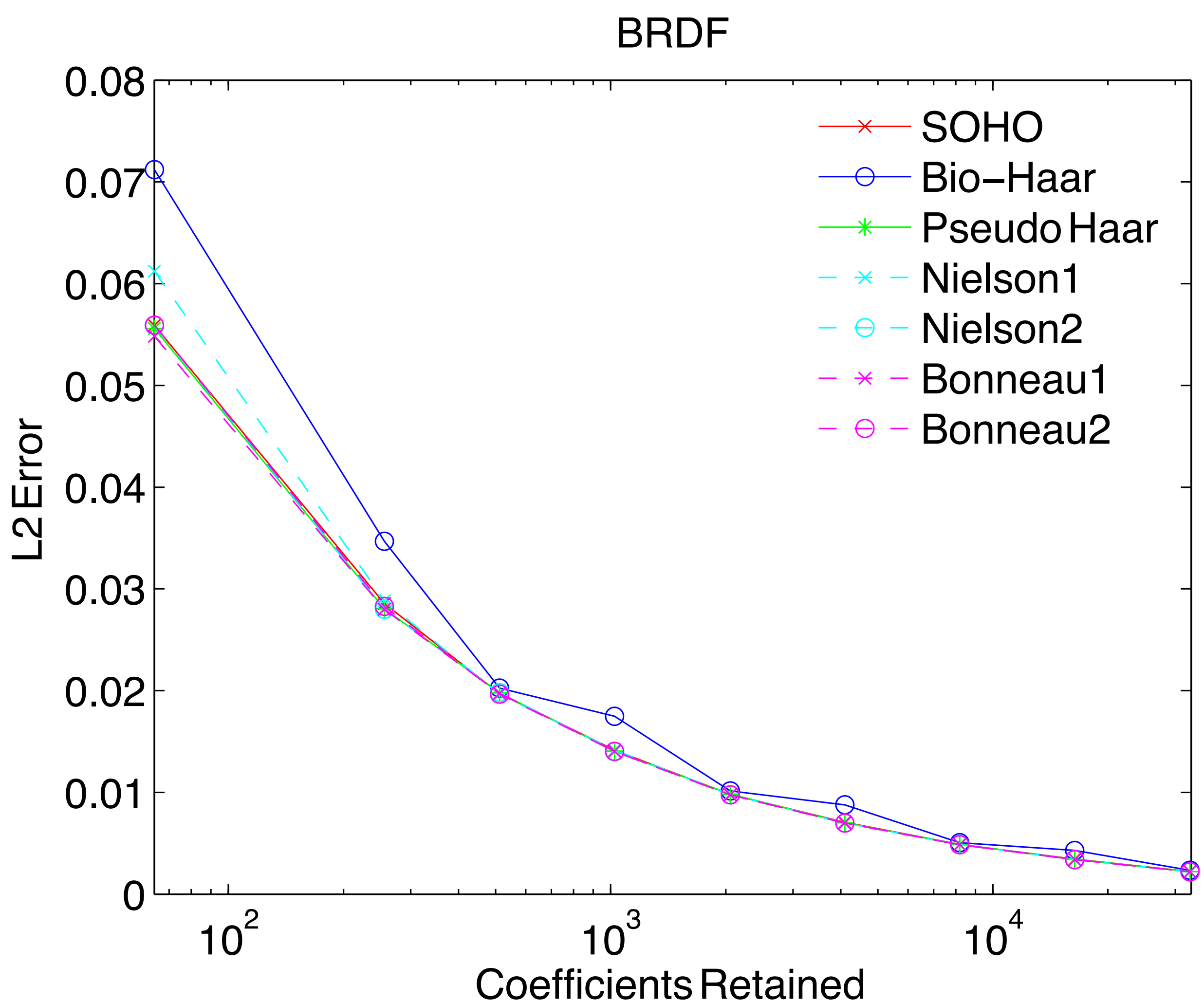

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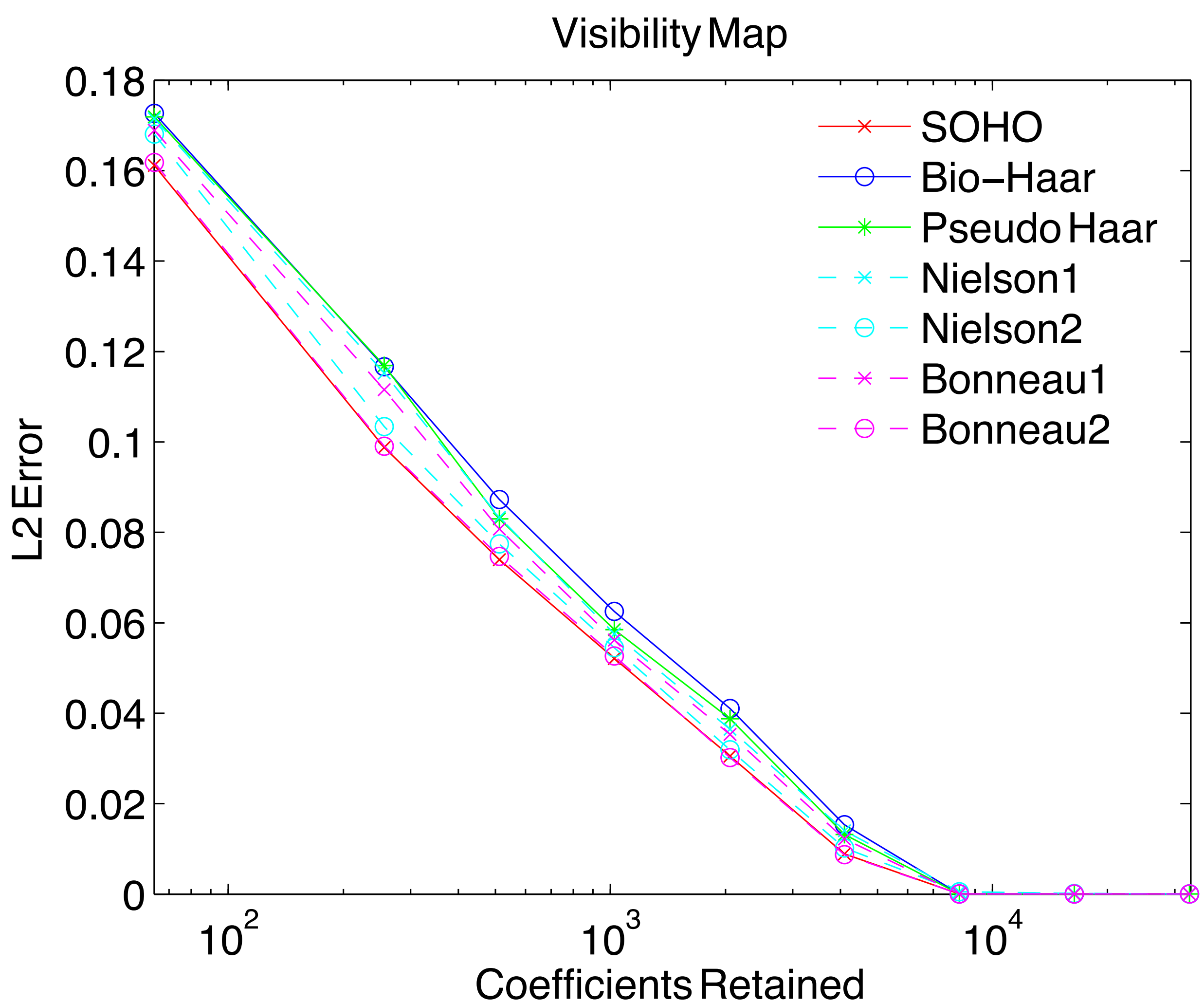

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Conclusion

- SOHO wavelet basis: symmetric and orthogonal spherical Haar wavelets.
- Nearly orthogonal spherical Haar wavelets are equivalent to SOHO basis for approximation.

Future Work

- Other orthogonal and symmetric spherical Haar wavelets?
- Orthogonal and symmetric spherical wavelets which are smooth?
- How efficient are nearly orthogonal wavelets for processing signals?
- How important is symmetry?
- Applications?

More details and Matlab code:

www.dgp.toronto.edu/people/lessig/soho

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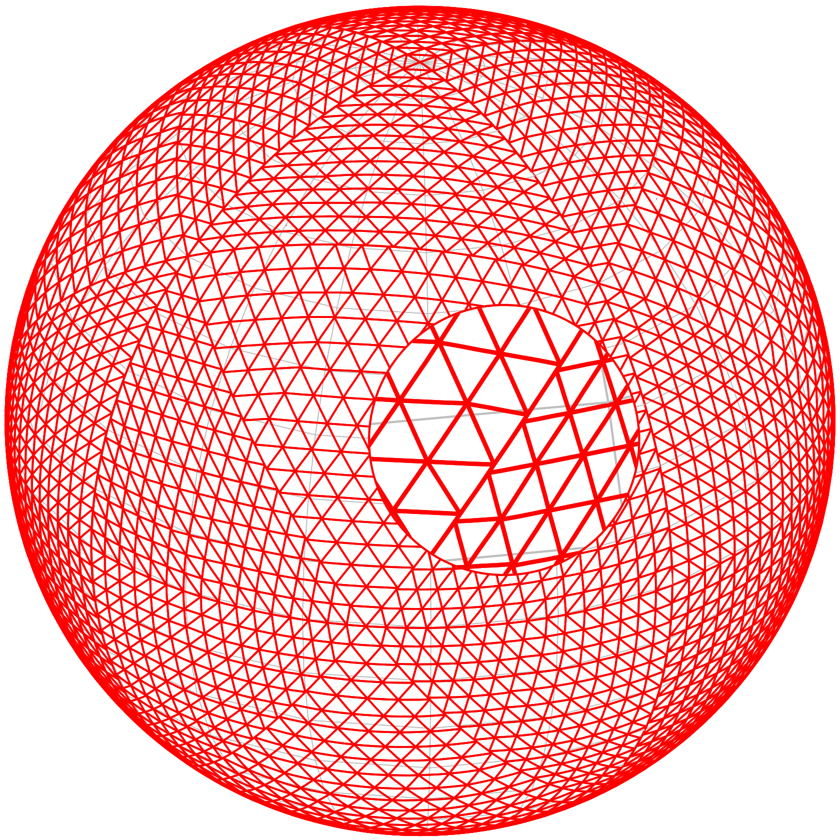
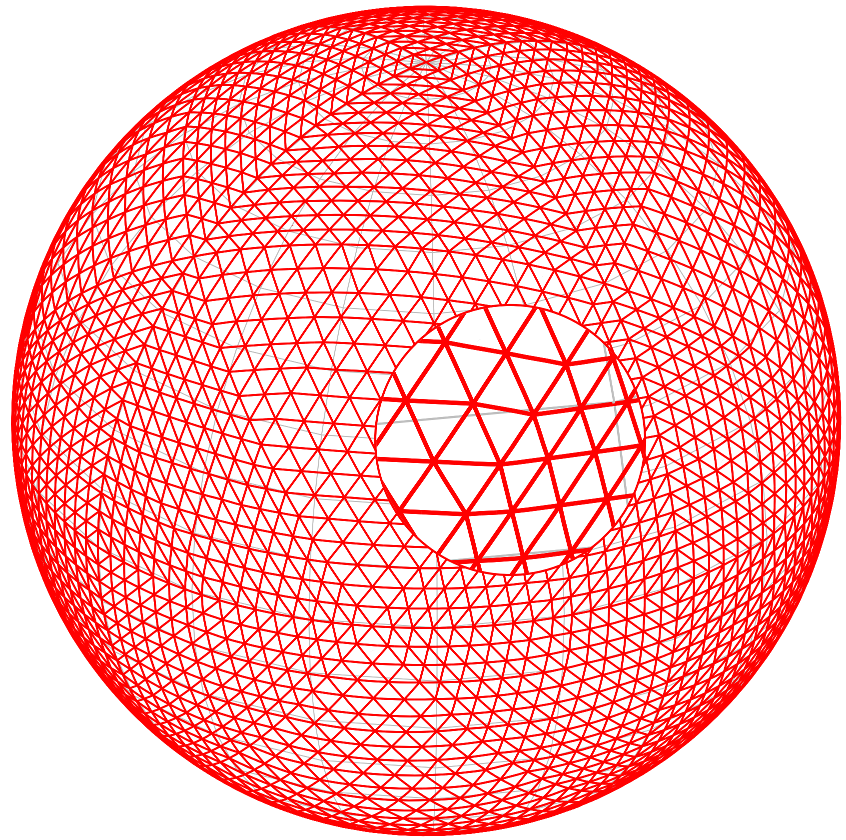
Appendix A: Subdivision Scheme



Our subdivision



Geodesic Bisector

Appendix B: SH versus SOHO