Spatio-Spectral Concentration on the Sphere

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Spatio-spectral concentration theory is concerned with the optimal space localization of a signal bandlimited in the frequency domain. The question was first considered in a series of classic papers by Slepian, Pollak, and Landau [1961; 1961; 1962; 1978] who studied the problem on the real line; an introduction to the early work on the subject can be found in two articles by Slepian [1976; 1983]. Recently, Simons and co-workers [2006; 2009; 2010b] extended these results to the sphere for functions bandlimited in the Spherical Harmonics domain. In the following, we will fix notation and then provide an introduction to the spatio-spectral concentration problem on the sphere. Our presentation will closely follow the exposition by Simons et al. [2006].

Preliminaries

Spherical Harmonics are the eigenfunctions of the Laplace-Beltrami operator $\Delta(\mathcal{S}^2)$ on the sphere and hence the analogues of the Fourier basis functions on $\mathcal{S}^2$. The eigenvalues $\sigma_l = l(l+1)$ of $\Delta(\mathcal{S}^2)$ have multiplicity $2l+1$ and (Legendre) Spherical Harmonics

$$y_{lm}(\theta, \phi) = \eta_{lm}\, P_{lm}(\cos\theta) \begin{cases} \sin(|m|\,\phi) & m < 0 \\ 1 & m = 0 \\ \cos(m\,\phi) & m > 0 \end{cases} \tag{1}$$

for $-l \leq m \leq l$ provide bases for the $(2l+1)$-dimensional eigenspaces $\mathcal{H}_l$. The $P_{lm}(t)$ in Eq. 1 are associated Legendre polynomials and η_{lm} is a normalization constant chosen such that the y_{lm} are orthonormal. The union $\bigoplus \mathcal{H}_l$ of all bands $l = 1 \ldots \infty$ is the Hilbert space $L_2(\mathcal{S}^2)$, defined as usual with inner product $\langle f, g \rangle = \int_{\mathcal{S}^2} f\, g\, d\omega$ for any $f, g \in L_2(\mathcal{S}^2)$. An orthonormal basis for the space is thus given by all Spherical Harmonics y_{lm} with $l = 1 \ldots \infty$ and $-l \leq m \leq l$. The well-known translational invariance of the Fourier basis corresponds to the rotational invariance of the spaces $\mathcal{H}_l$: $f_l \in \mathcal{H}_l$ implies $\bar{f}_l = \mathrm{R}\, f_l \in \mathcal{H}_l$ where $\mathrm{R} \in \mathrm{SO}(3)$ is an arbitrary rotation and the action of the rotation group $\mathrm{SO}(3)$ is defined point-wise as $\mathrm{R} f = f \circ \mathrm{R}^{-1}$. An important result in the theory of Spherical Harmonics is the addition theorem

$$P_l(\cos\gamma) = \frac{4\pi}{2l+1} \sum_{m=-l}^{l} y_{lm}(\bar{\omega})\, y_{lm}(\omega), \tag{2}$$

where $P_l(t) = P_{l0}(t)$ is the Legendre polynomial of degree l and $\gamma = \angle(\bar{\omega}, \omega)$. Eq. 2 provides an expansion of the reproducing kernel (or Zonal Harmonic) $P_l(\cos\gamma) = P_l(\bar{\omega} \cdot \omega)$ for $\mathcal{H}_l$ in Spherical Harmonics in a fixed coordinate frame.

We refer to the books by Freeden and co-workers [Freeden et al. 1998; Freeden and Schreiner 2009] for a more detailed discussion of the theory of Spherical Harmonics and proofs for our claims.

Spatio-Spectral Concentration on the Sphere

Many applications require the efficient representation of signals defined over a subset $\mathcal{U} \subset \mathcal{S}^2$ of the sphere. Compact representations are obtained with functions localized in space while bandlimited expansions provide the advantages of Spherical Harmonics such as rotational invariance. However, the analogue of the Fourier uncertainty principle [Daubechies 1992; Mallat 1999] for the sphere [Freeden et al. 1998, Theorem 5.5.1] shows that both properties are incompatible and no bandlimited representation can be localized in space. To retain the advantages of Spherical Harmonics while being able to efficiently represent spatially localized signals we seek bandlimited functions $g \equiv g_L = \sum_{lm} g_{lm}\, y_{lm} \in \mathcal{H}_{\leq L}$ which maximize the concentration measure

$$\lambda = \frac{\|g\|_{\mathcal{U}}^2}{\|g\|_{\mathcal{S}^2}^2} = \frac{\int_{\mathcal{U}} |g|^2 d\omega}{\int_{\mathcal{S}^2} |g|^2 d\omega} \tag{3}$$

for arbitrary but fixed region $\mathcal{U} \in \mathcal{S}^2$. Expanding Eq. 3 in Spherical Harmonics yields

$$\lambda = \frac{\sum_{lm} \sum_{l'm'} g_{lm}\, g_{l'm'}\, d_{lm\,l'm'}}{\sum_{lm} g_{lm}} \tag{4}$$

where we defined

$$d_{lm\,l'm'} = \int_{\mathcal{U}} y_{lm}\, y_{l'm'}\, d\omega.$$

Eq. 4 is the spatio-spectral concentration problem on the sphere in the frequency domain. By re-arranging the basis function coefficients in vector form, g, and the restricted inner products $d_{lm,l'm'}$ as a matrix, D, it can be stated more concisely as

$$\lambda = \frac{\mathrm{g}^T \mathrm{D}\, \mathrm{g}}{\mathrm{g}^T \mathrm{g}}. \tag{5}$$

Eq. 5 is a matrix variational problem and it is known [Horn and Johnson 1990] that vectors g_i which render the problem stationary satisfy the eigenvalue equation

$$\mathrm{D}\, \mathrm{g}_i = \lambda_i \mathrm{g}_i. \tag{6}$$

In the spatial domain Eq. 6 becomes

$$\int_{\mathcal{U}} D(\bar{\omega}, \omega)\, g_i(\omega) d\omega = \lambda_i\, g_i(\bar{\omega}) \tag{7}$$

where the kernel $D(\bar{\omega}, \omega)$ is given by

$$D(\bar{\omega}, \omega) = \sum_{l=0}^{L} \sum_{m=-l}^{l} y_{lm}(\bar{\omega})\, y_{lm}(\omega) = \sum_{l=0}^{L} \frac{2l+1}{4\pi} P_l(\bar{\omega} \cdot \omega)$$

which is the reproducing kernel for $\mathcal{H}_{\leq L}$ but with the domain of integration restricted to $\mathcal{U}$, cf. Eq. 2.

The functions $g_i(\omega) \in \mathcal{H}_{\leq L}$ in Eq. 7 are known as spherical Slepian functions and it follows from the symmetry and positivity of $D(\bar{\omega}, \omega)$ that these can be chosen to be orthogonal over $\mathcal{U}$. Additionally, the $\{g_i\}$ form an orthonormal basis for $\mathcal{H}_{\leq L}$. Hence,

$$\int_{\mathcal{S}^2} g_i\, g_j\, d\omega = \delta_{ij} \qquad \int_{\mathcal{U}} g_i\, g_j\, d\omega = \lambda_i\, \delta_{ij}.$$

By Eq. 3, the eigenvalue λ_i provides a measure for the spatial concentration of the Slepian function g_i and by construction g_j is the maximally concentrated function in $\mathcal{H}_{\leq L}$ which is orthogonal to all g_i with $i < j$. It follows from the Fourier uncertainty principle and the properties of $D(\bar{\omega}, \omega)$ that the eigenvalues satisfy $1 > \lambda_i \geq \cdots \geq \lambda_n > 0$ with $n = (L+1)^2$.

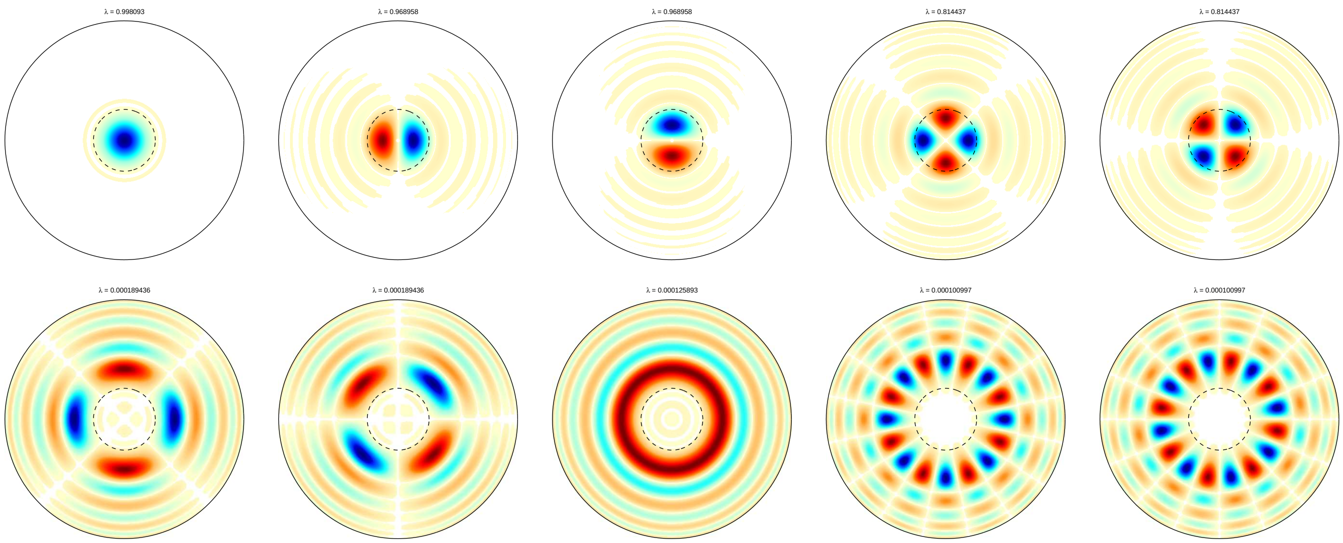

Fig. 1. Spherical Slepian functions for $L = 20$ and a spherical cap with $\theta \leq 15^\circ$ (dotted circle). The first row shows the first five Slepian functions corresponding to the five largest eigenvalues and the second row Slepian functions with the corresponding eigenvalues very close to zero. Positive values are shown in blue and negative values in red.

The spectrum of the spatio-spectral concentration problem has a characteristic shape: The first N eigenvalues are close to unity, followed by a region of exponential decay from one to zero, and the remaining eigenvalues are negligible. For the real line it has been shown [Slepian 1965; Landau 1965; Landau and Widom 1980] that for a bandlimit W and a region of concentration T the number of eigenvalues greater than $\epsilon \approx 0$ satisfies

$$N(\epsilon) = \frac{W\,T}{2\pi} c + \left(\log \left(\frac{1-\epsilon}{\epsilon} \right) \frac{\mu\,\nu}{\pi^2} \right) \log c + o(\log c) \qquad (8)$$

where c is a scaling parameter for the spatio-spectral region of concentration and the three terms in Eq. 8 correspond to the three characteristic parts of the spectrum. In the second summand μ and ν are the total numbers of intervals in the frequency and time domain which together have size W and T, respectively, and they measure the size of the boundary of the region of concentration. The first term in Eq. 8 is known as the Shannon number N and we will refer to the first two terms as generalized Shannon number N_g. Unfortunately, for the sphere no results analogous to Eq. 8 have been established to date and the rather technical nature of the proof by Landau and Widom [1980] prevents a straight forward extension to $\mathcal{S}^2$. However, the analogy to the real line and a two-dimensional setting studied by Slepian [1964] as well as experimental results [Simons et al. 2006; Simons 2010a] suggests that

$$N(\epsilon) = \frac{C}{4\pi} + \log \left(\frac{1-\epsilon}{\epsilon} \right) \mathrm{B}(\partial\mathcal{U}) \log(C) + o(\log C) \qquad (9)$$

where $\mathrm{B}(\partial\mathcal{U})$ is a function which depends on the boundary $\partial\mathcal{U}$ of the region of concentration and $C = (L+1)^2 A(\mathcal{U})$, cf. Fig. 2. The first summand of Eq. 9 is again known as the (spherical) Shannon number and it is the dominant term of $N(\epsilon)$ for a sufficiently large region of spatio-spectral concentration. The spectrum is then well described by a step function with the discontinuity at N and a function localized in $\mathcal{U}$ can be represented accurately with only the first N Slepian functions which are all well confined in the region of concentration, cf. Fig. 1. If the spatio-spectral region of concentration is small then a non-negligible contribution to the total spectrum is contained in the transition region from one to zero. In this case, the number of Slepian functions which is needed to accurately represent a localized signal is given by the generalized Shannon number N_g and because the Slepian basis functions are no longer well concentrated the signal representation will have significant leakage into the complement $\mathcal{S}^2 \setminus \mathcal{U}$ of the region of concentration $\mathcal{U}$.

The problem dual to optimally concentrating a bandlimited function in a region $\mathcal{U} \subset \mathcal{S}^2$ is optimally concentrating a signal in the frequency domain which is localized in $\mathcal{U} \subset \mathcal{S}^2$. Interestingly, the solution to both problems is (essentially) identical. We refer to the paper by Simons et al. [2006] for details.

Computation of Slepian Functions

The space $\mathcal{H}_{\leq L}$ of bandlimited functions on the sphere is finite dimensional. Spherical Slepian functions can thus be obtained by solving Eq. 6 using standard numerical algorithms; in practice some care is however required since the eigenvalues are clustered and the computation of the eigenvectors is hence numerically delicate. In the case of a polar spherical cap the computations are simplified by the existence of a tri-diagonal differential operator which commutes with $D(\omega, \bar{\omega})$ and which has a well-behaved spectrum. We refer to [Simons et al. 2006] and references therein for details.

Applications

Slepian functions have found applications in a variety of fields such as optics, signal processing, electromagnetism, acoustics, and medical imaging. We believe they will also prove useful in computer graphics. For example, Slepian functions are well-suited for representing signals concentrated in space and frequency and might hence be employed in rendering where functions defined over the hemisphere or a subset thereof are prevalent and bandlimited signal expansions have a long history. In various areas, Slepian functions have also been used for the estimation of signals and their power spectra from noisy and incomplete observations. This suggests their use for instance in the processing of range scanner data.

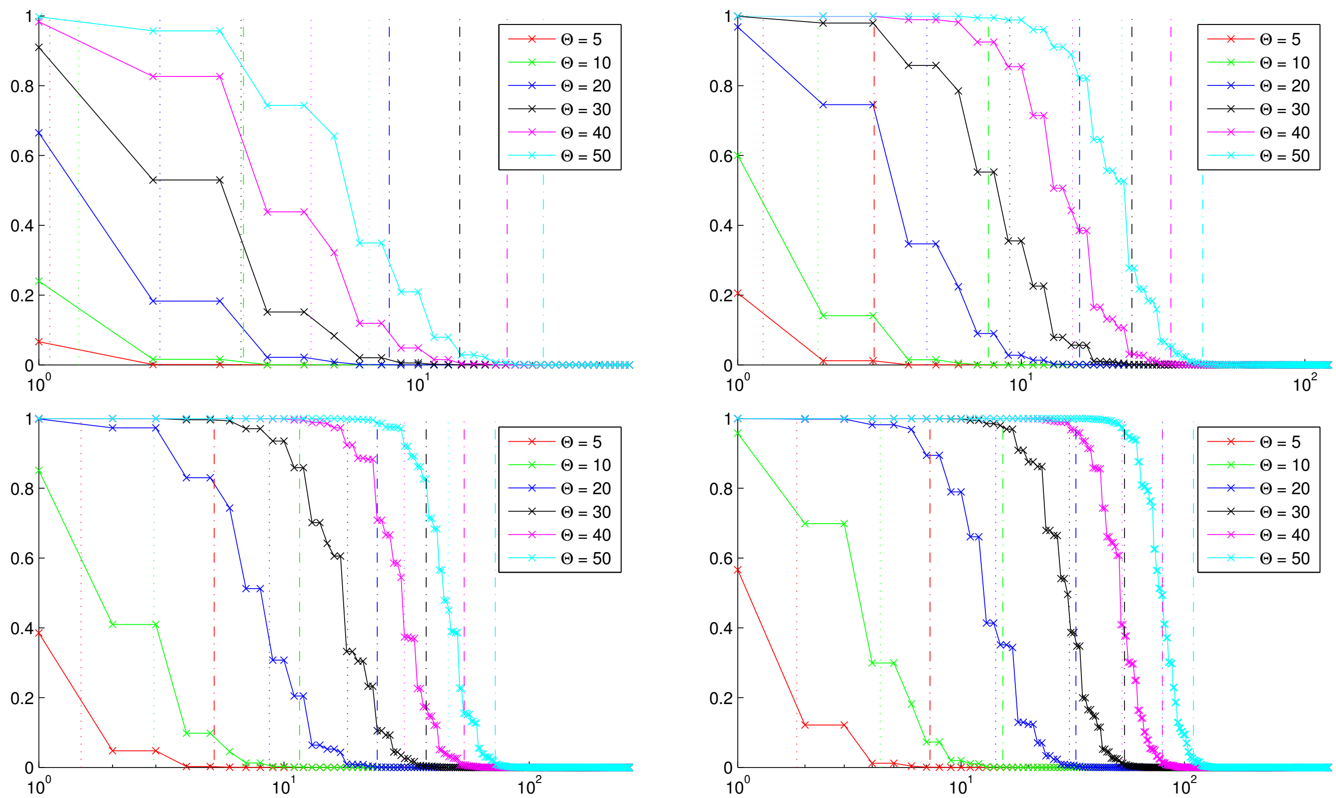

Fig. 2. Spectrum of the spatio-spectral concentration problem on the sphere for spherical caps with $\theta \leq \Theta$ and $L = 5, 10, 15, 20$ (left to right, top to bottom). The eigenvalue index is shown on the X axis and the magnitude $|\lambda_i|$ of the eigenvalues on the Y axis. Shown are also the Shannon number N (dotted) and generalized Shannon number N_g (dash-dotted), the latter one computed with $\mathrm{B}(\partial U) = \lg\left((L+1)^2 |\partial \mathcal{U}|^2\right) / \lg\left(2\pi\right)$. Clearly visible is the importance of the second term in N_g when the region of spatio-spectral concentration is small.

ACKNOWLEDGMENTS

The software used to generate the figures, which is based on code made available by Frederik Simons, http://geoweb.princeton.edu/people/simons/software.html, is available at http://www.dgp.toronto.edu/people/lessig/slepian/.

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