# On the Effective Dimension of Light Transport

Christian Lessig and Eugene Fiume

Dynamic Graphics Project, Department of Computer Science, University of Toronto

# Problem statement

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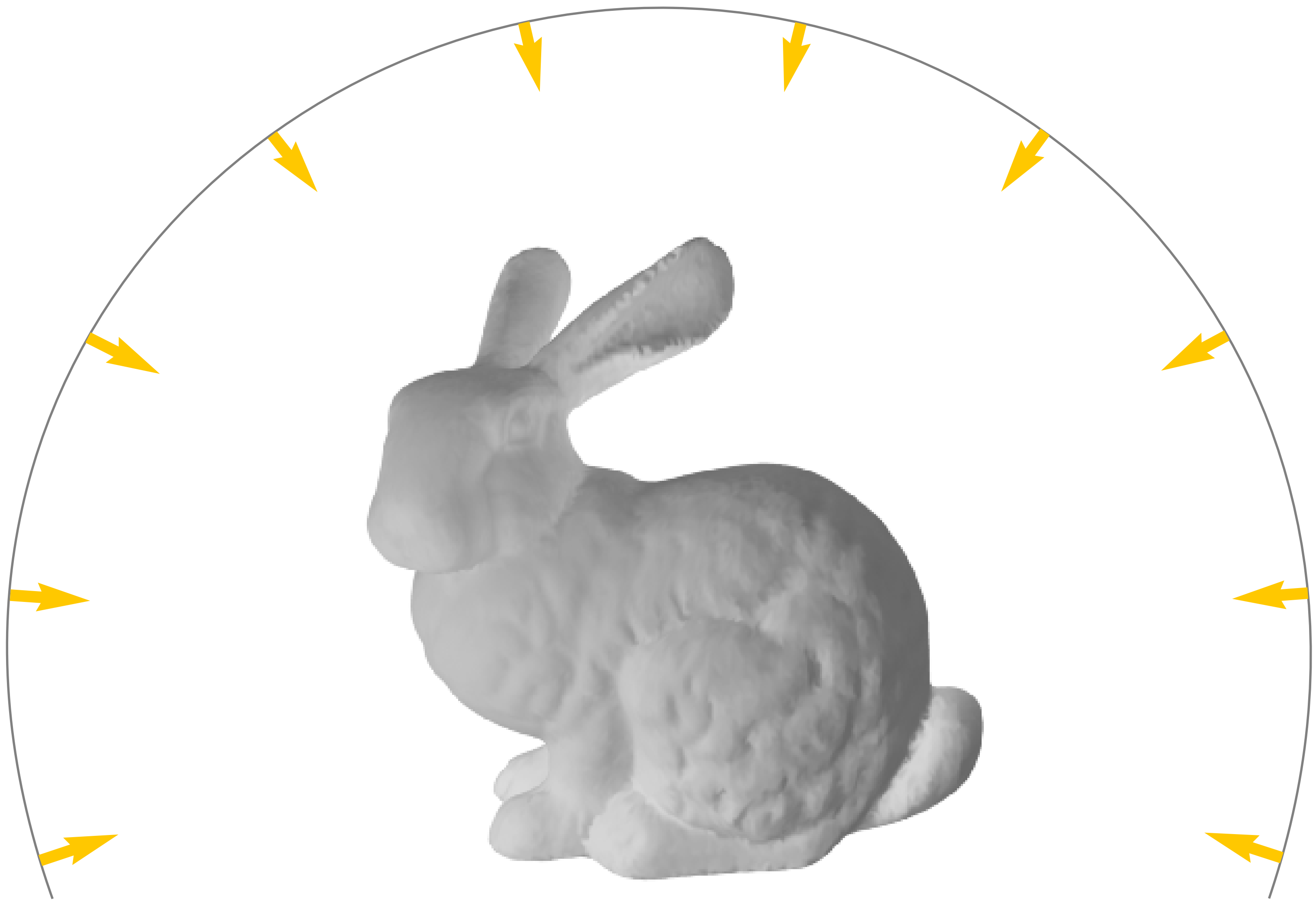

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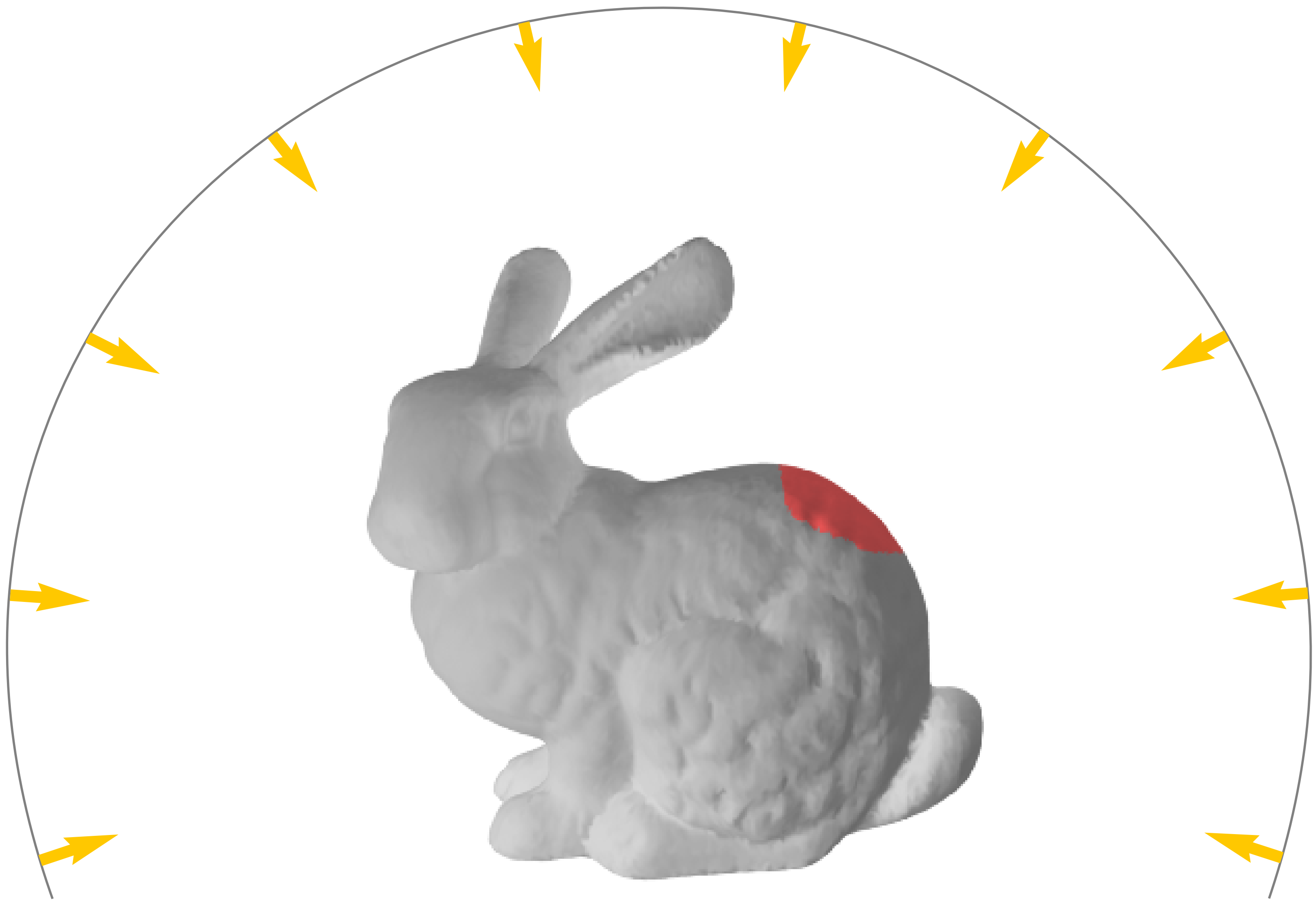

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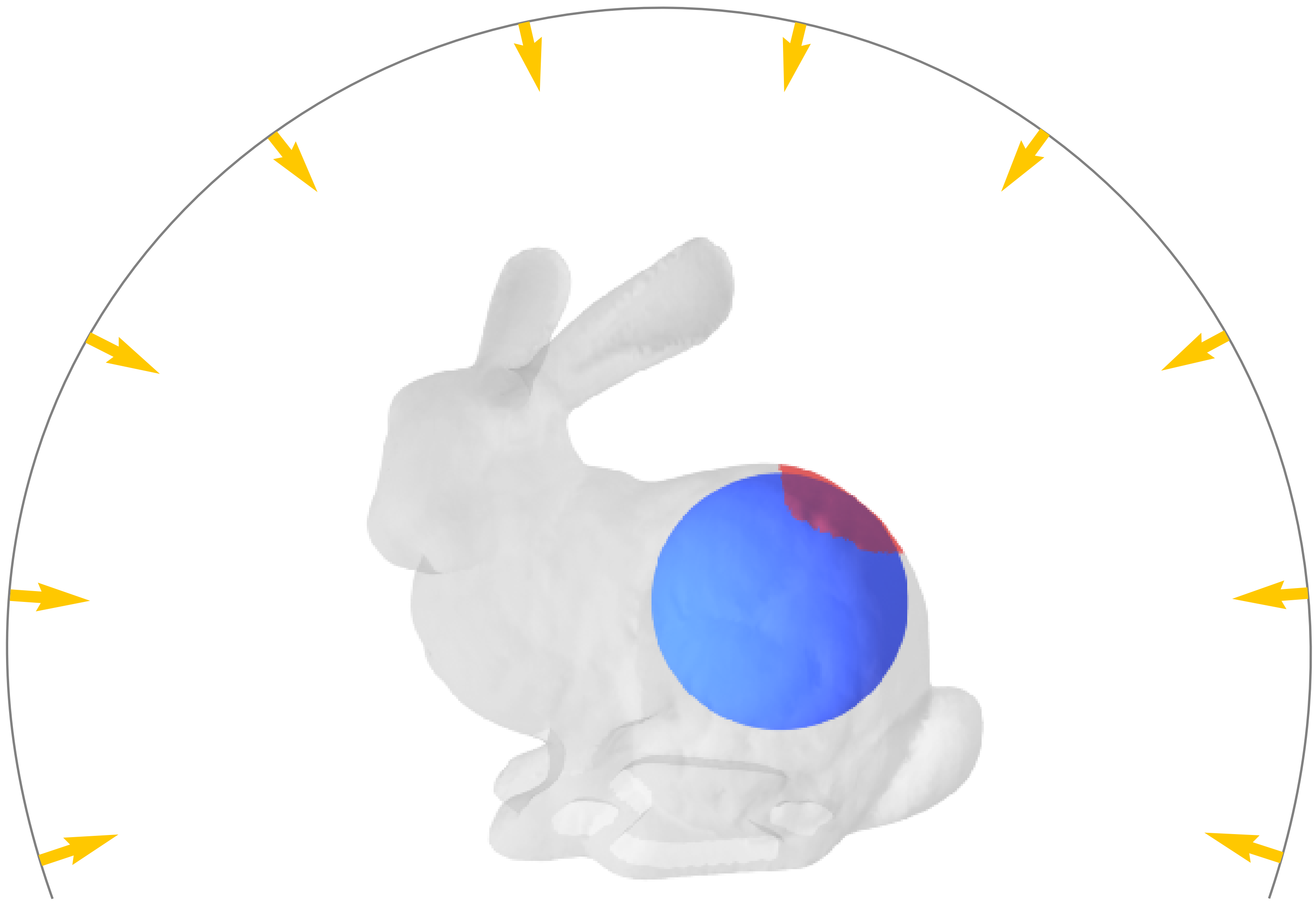

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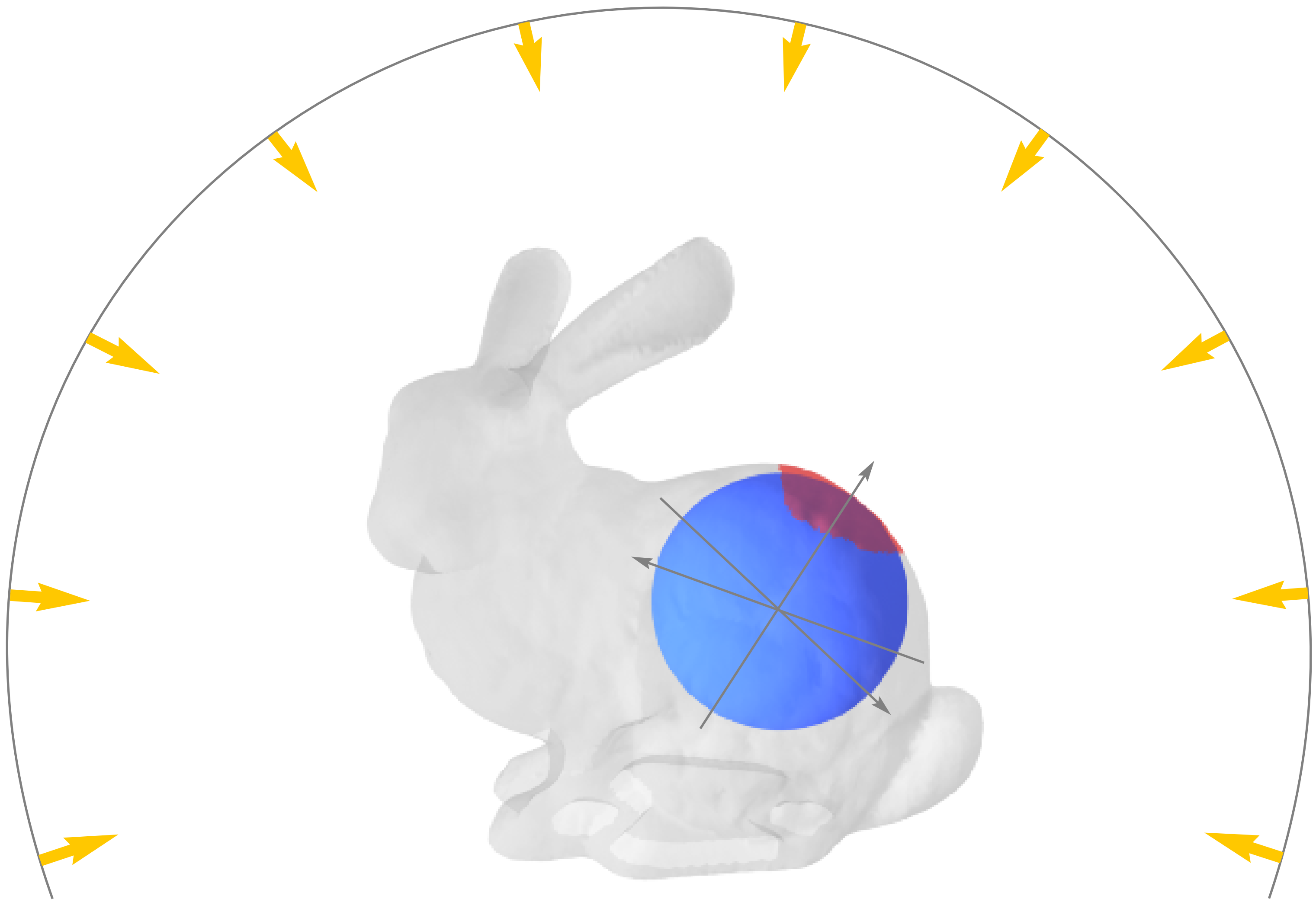

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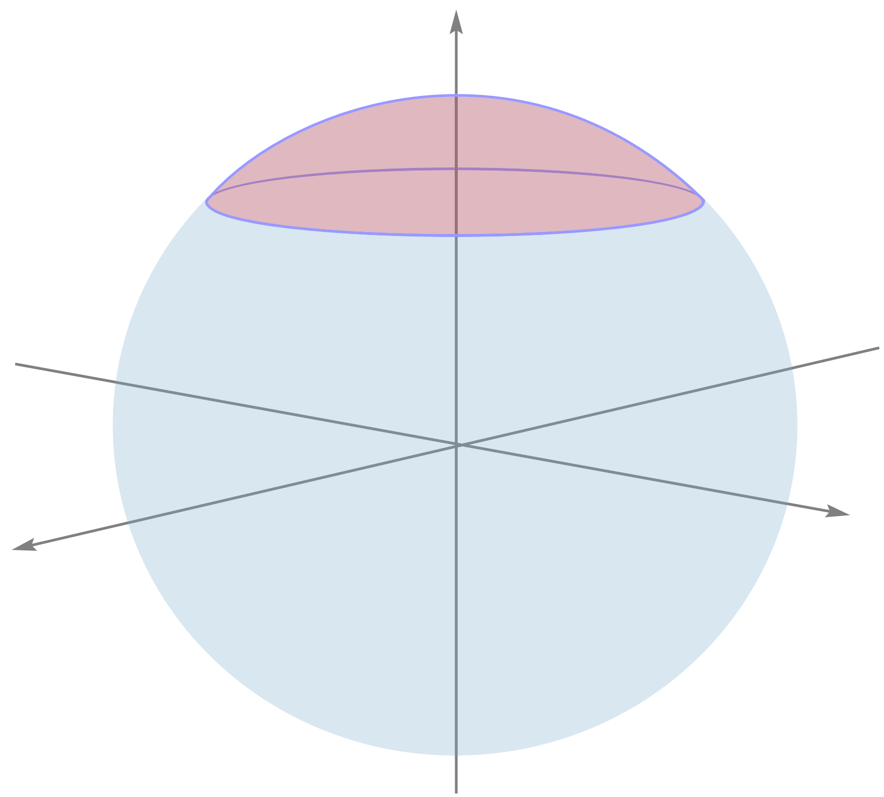

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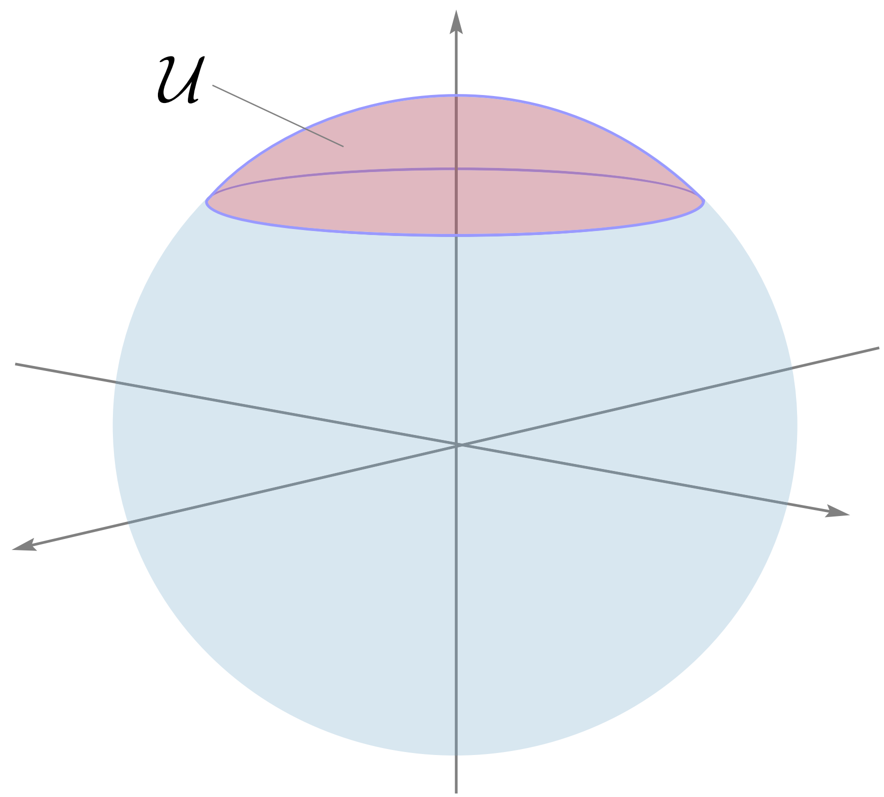

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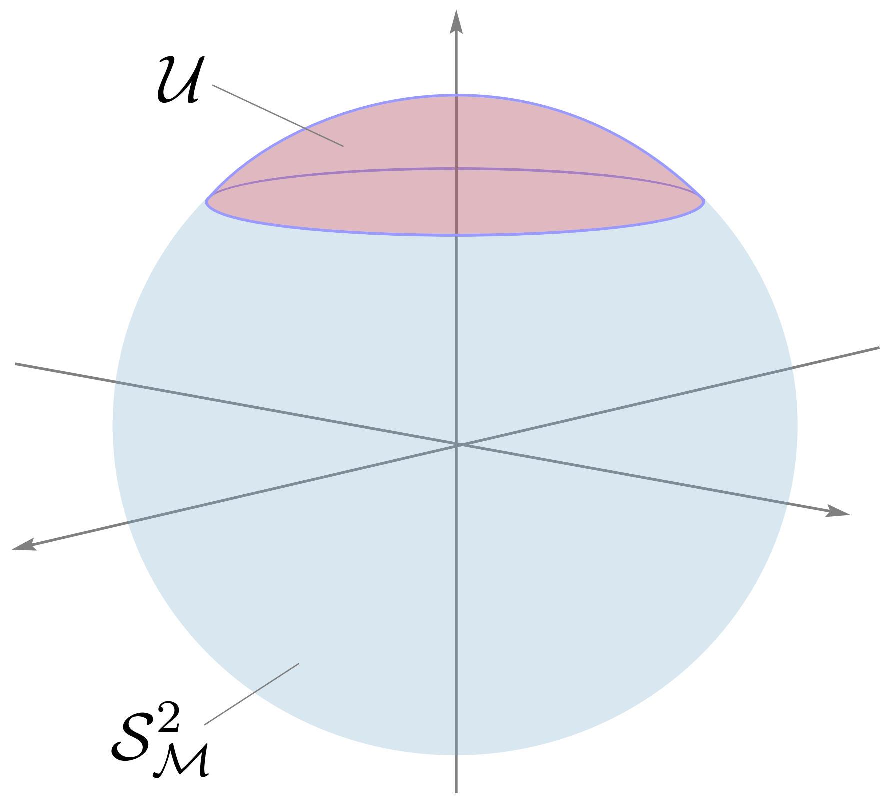

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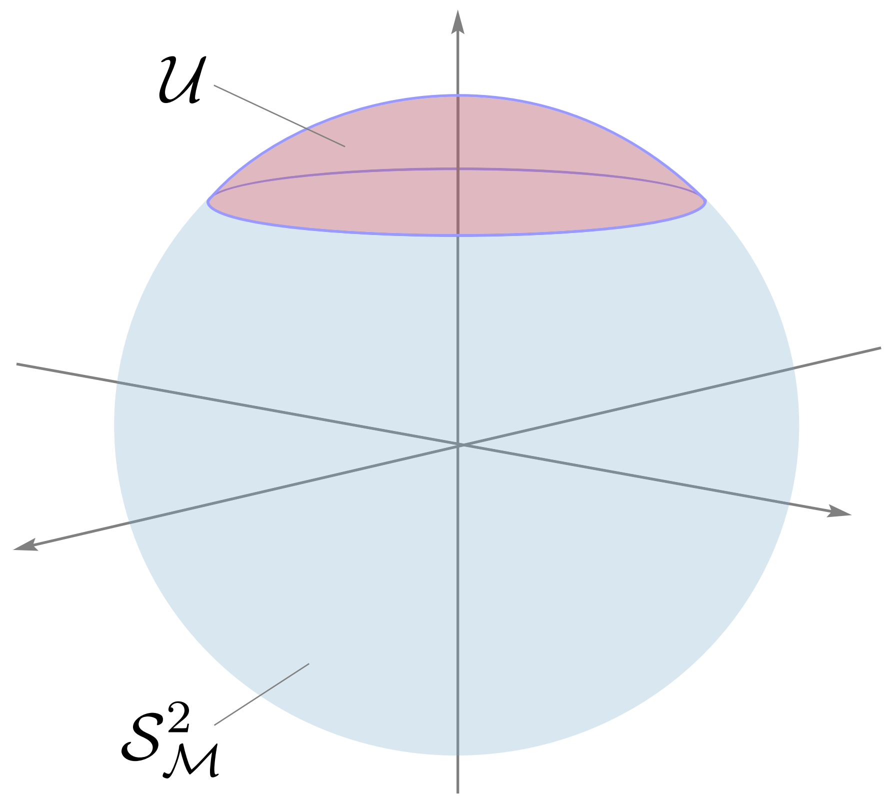
Shading equation:

$$B(x) = \int_{\mathcal{H}^2} T(\mathbf{n}(x) \cdot \omega)\, E(\omega)\, d\omega$$



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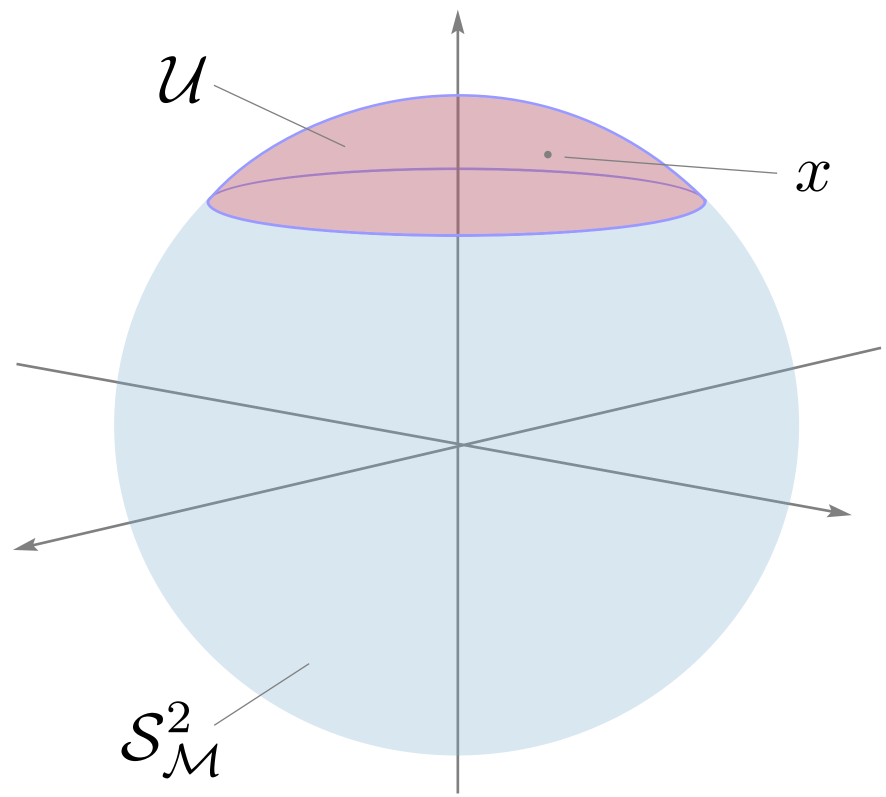

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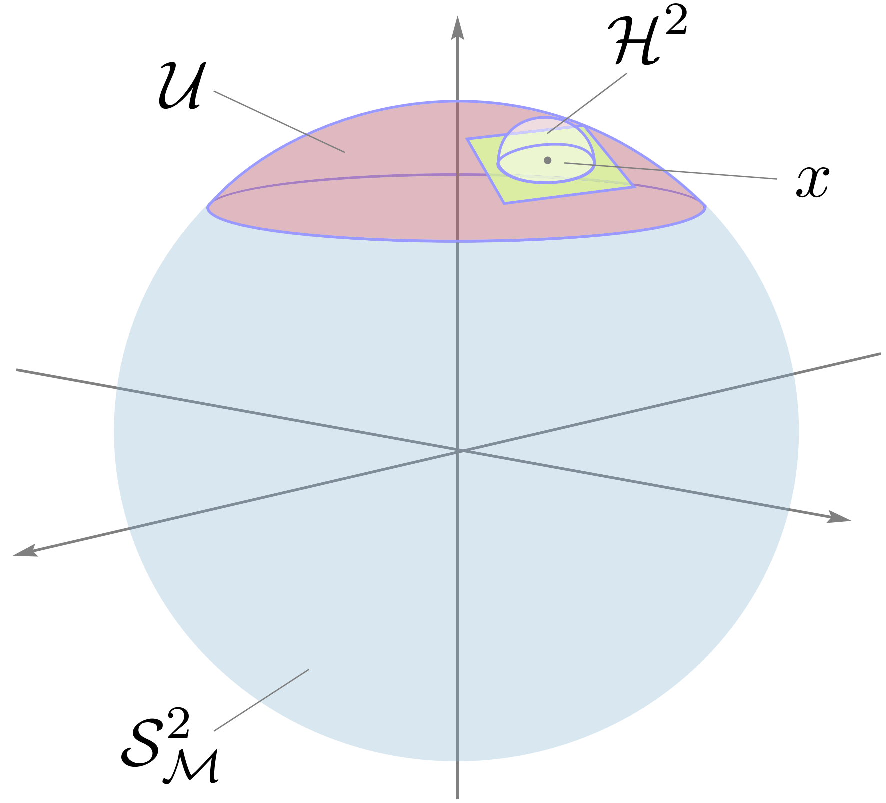

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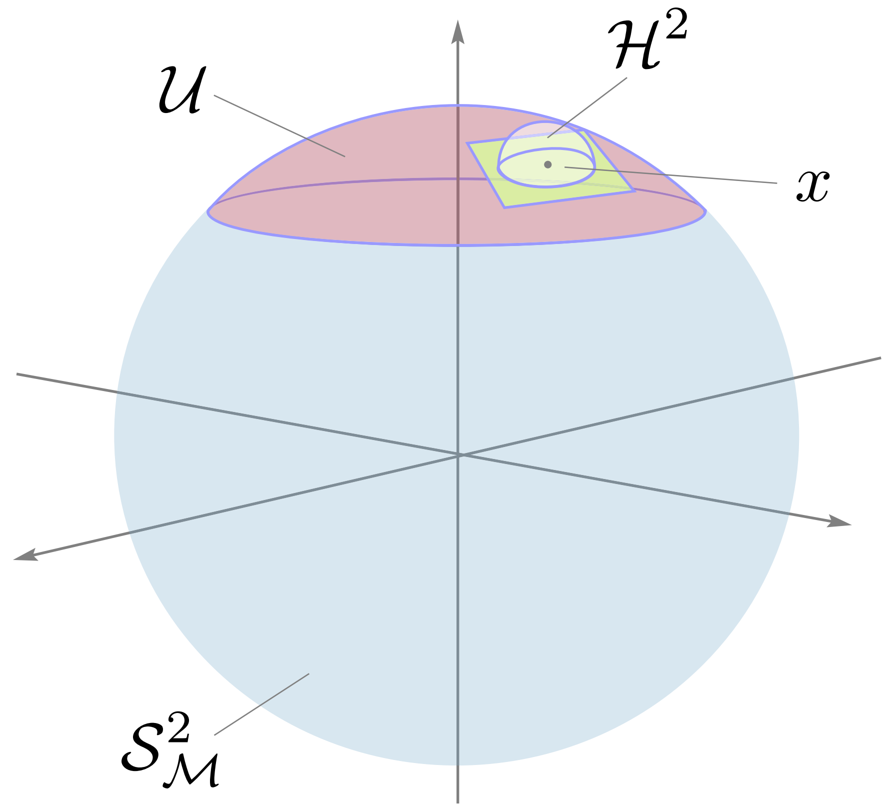
$$\|\mathrm{B} - \tilde{\mathrm{B}}\|_{\mathcal{U}}^2 = \|\mathrm{TE} - \tilde{\mathrm{T}}\mathrm{E}\|_{\mathcal{U}}^2 \approx \epsilon$$



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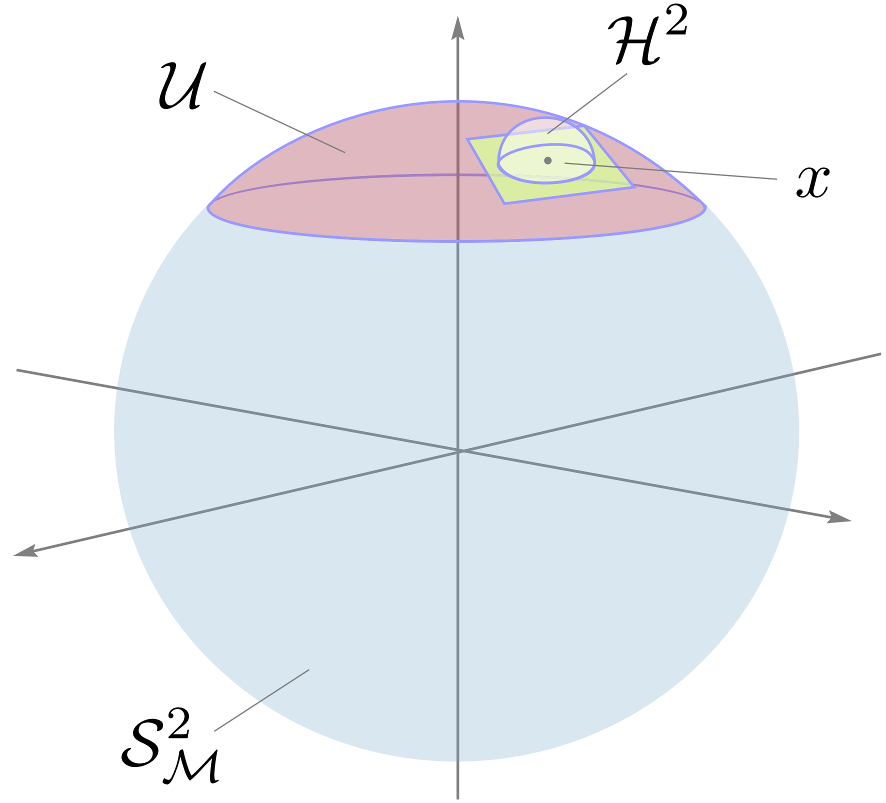

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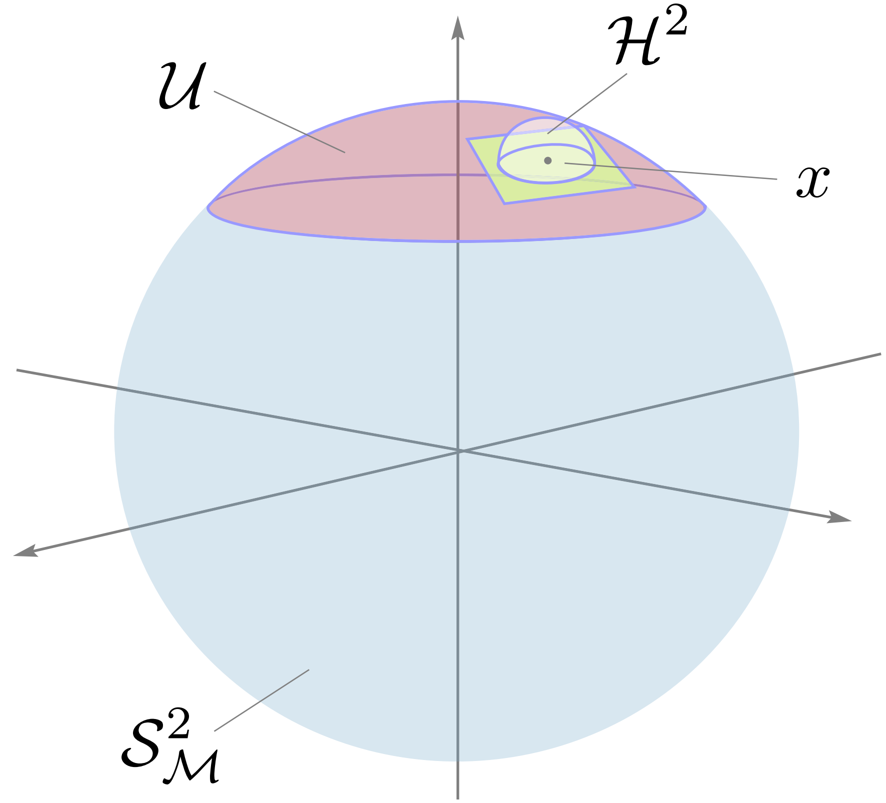
effective dimension



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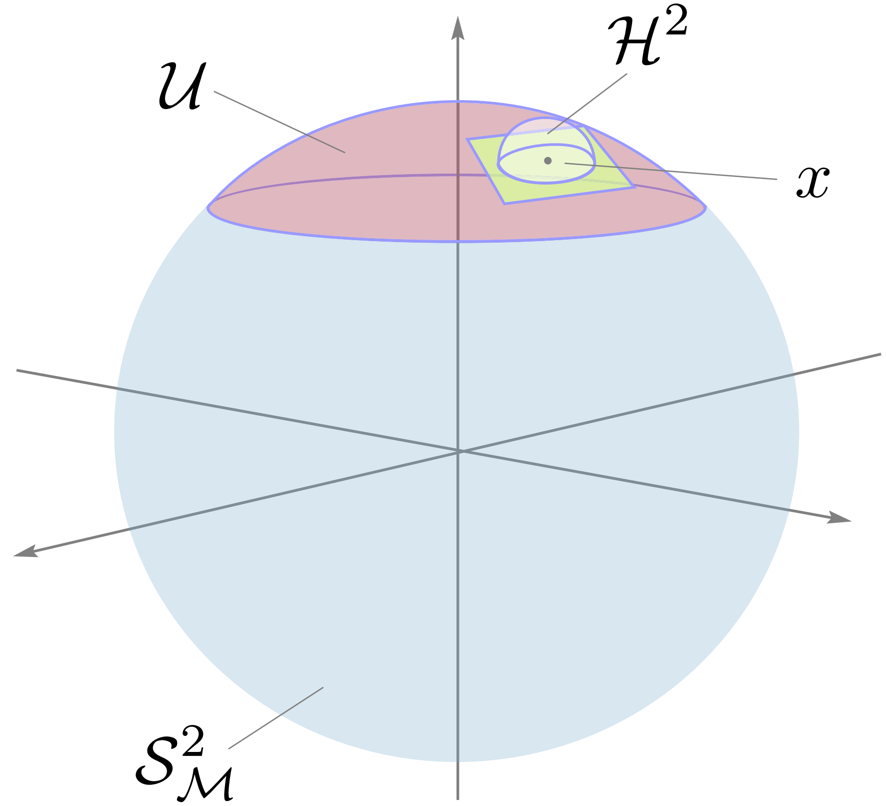

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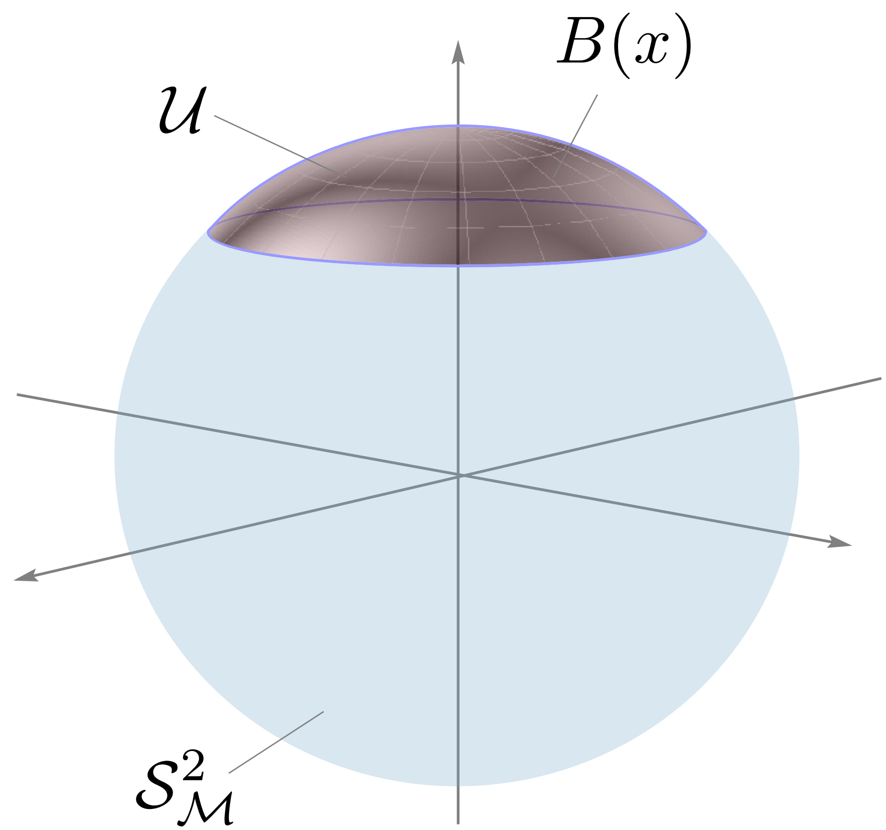

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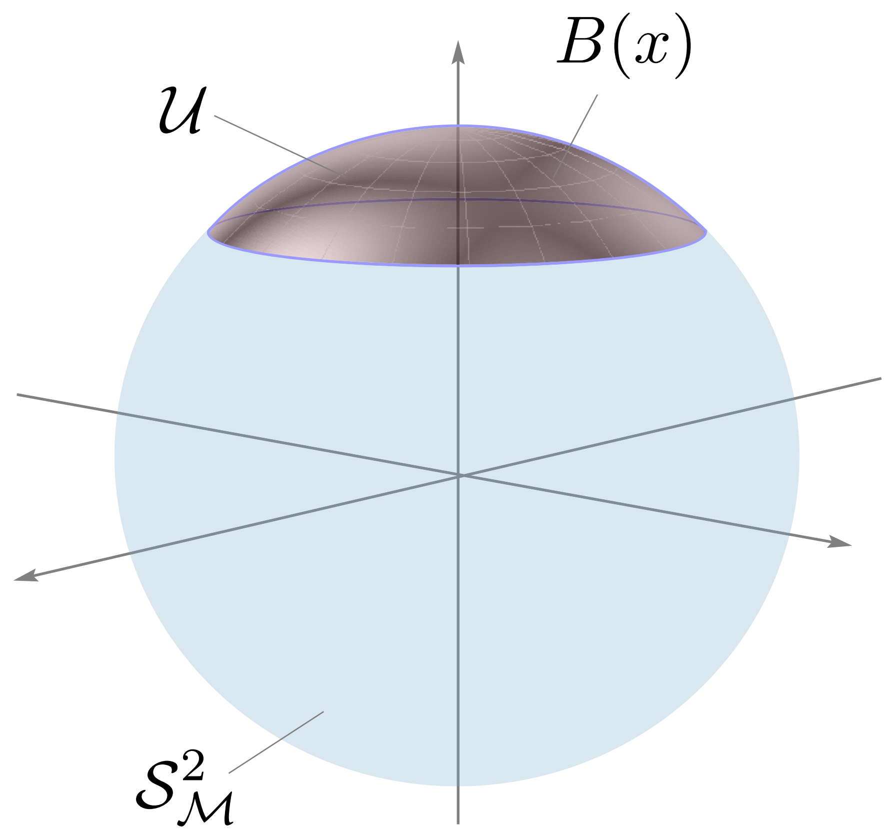
but $\mathbf{n}(x) \simeq \omega \in \mathcal{S}^2_{\mathcal{M}}$



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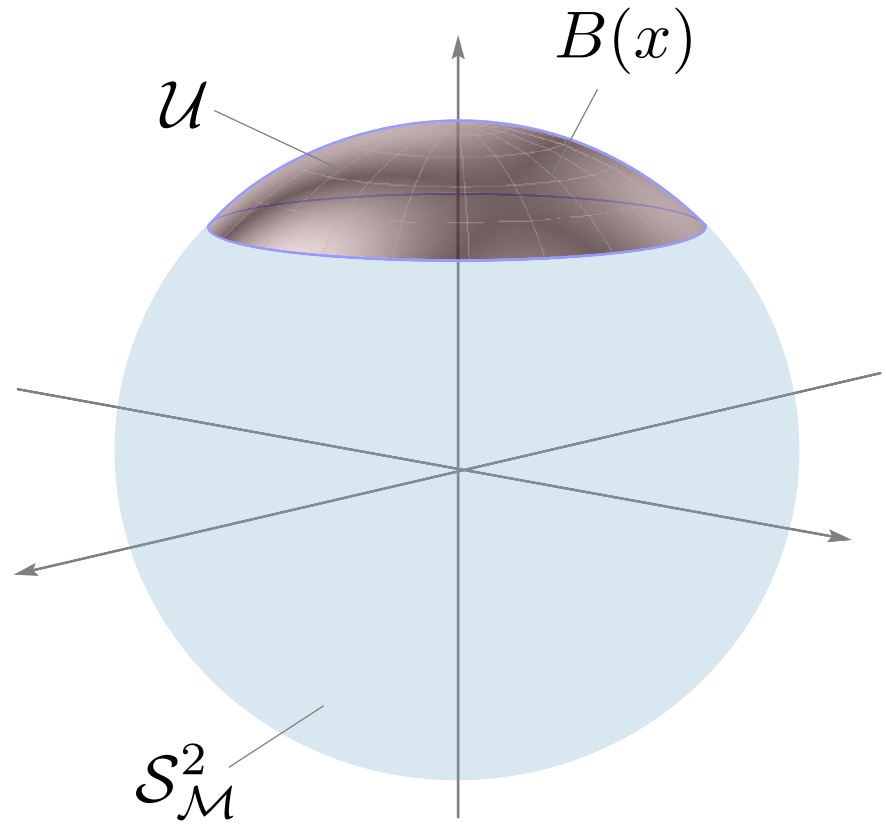

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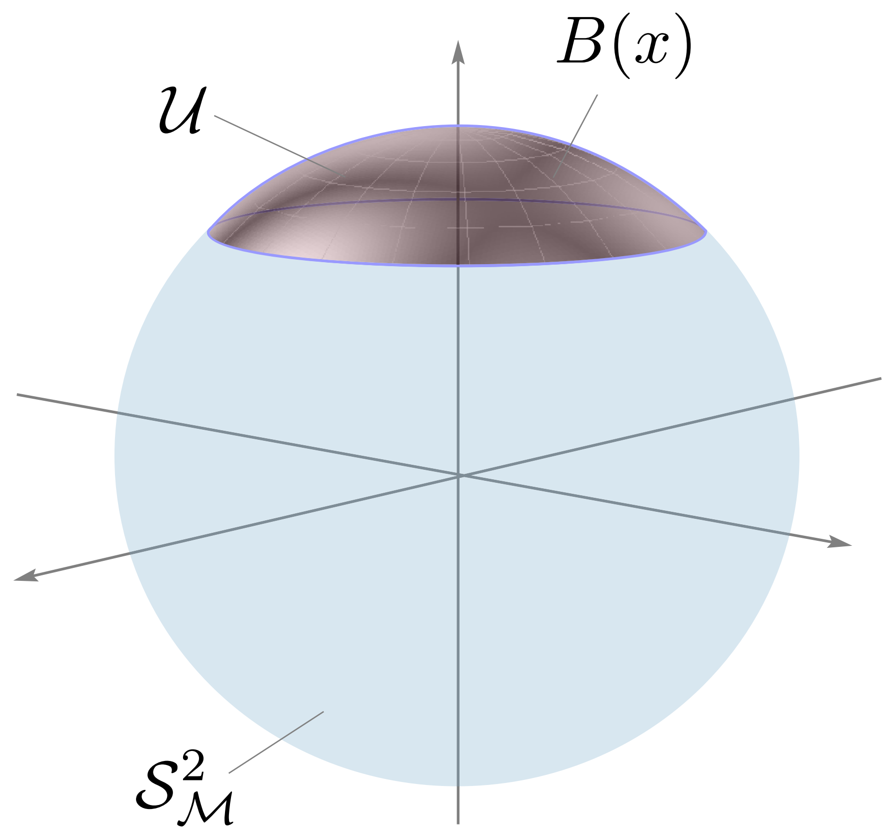

# Approximation

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with *effective dimension* $K \ll N$ such that

$$B(\omega) \approx \tilde{B}(\omega) = \sum_{i=1}^{K} \tilde{b}_i \, \varphi_i$$

# Approximation

## Approximation error

$$\left\| B(\omega) - \tilde{B}(\omega) \right\|_{\mathcal{U}}^{2} = \sum_{i=K+1}^{N} \tilde{b}_i \left\| \varphi_i(\omega) \right\|_{\mathcal{U}}^{2}$$

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which, for arbitrary input signals, is minimized if

$$\| \varphi_i(\omega) \|_{\mathcal{U}}^2 \quad , \quad i = K+1 \ldots N$$

is minimal.

# Spatio-Spectral Concentration Theory

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Objective: Extremize concentration measure[1]

$$\lambda = \frac{\|g\|_{\mathcal{U}}^2}{\|g\|_{\mathcal{S}^2}^2} = \frac{\int_{\mathcal{U}} g^2 d\omega}{\int_{\mathcal{S}^2} g^2 d\omega} \quad , \quad g \in \mathcal{H}_{\leq L}$$

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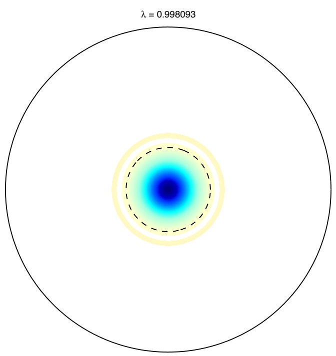

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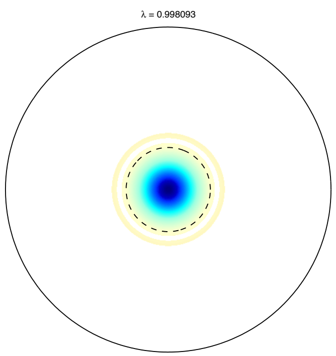


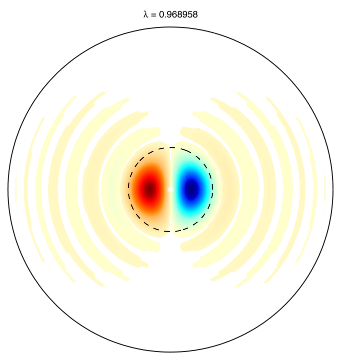

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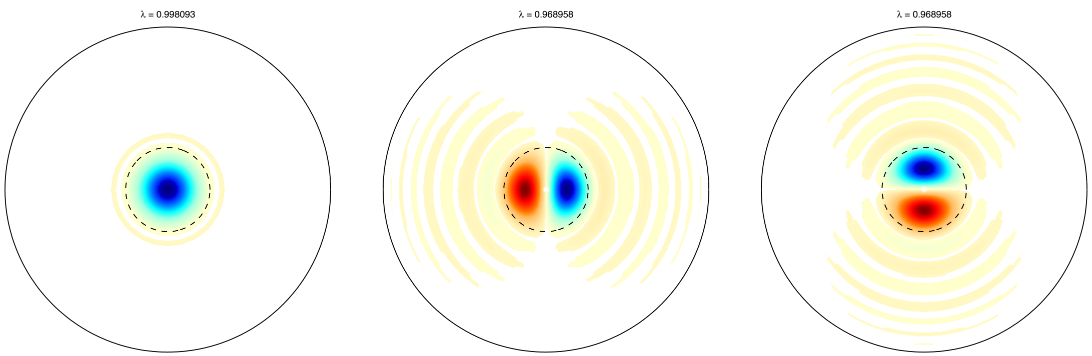

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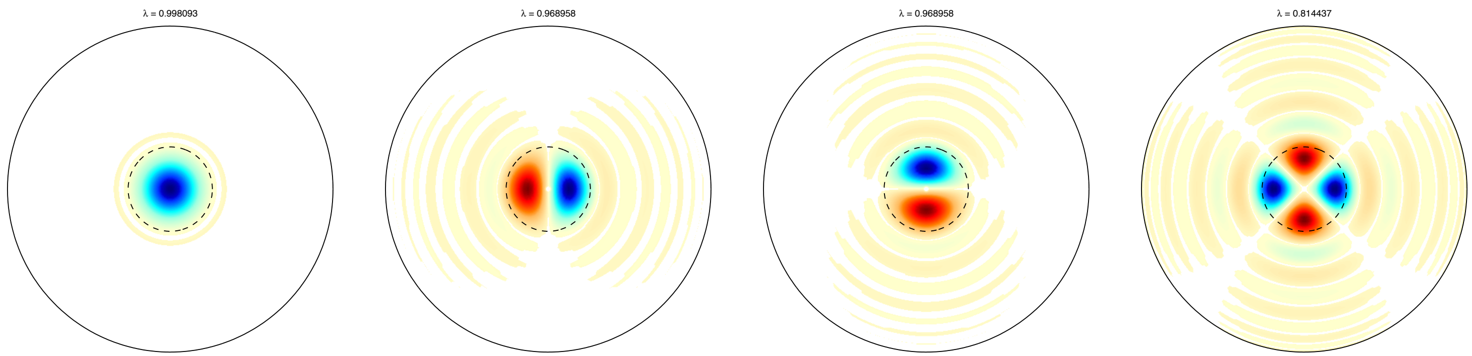

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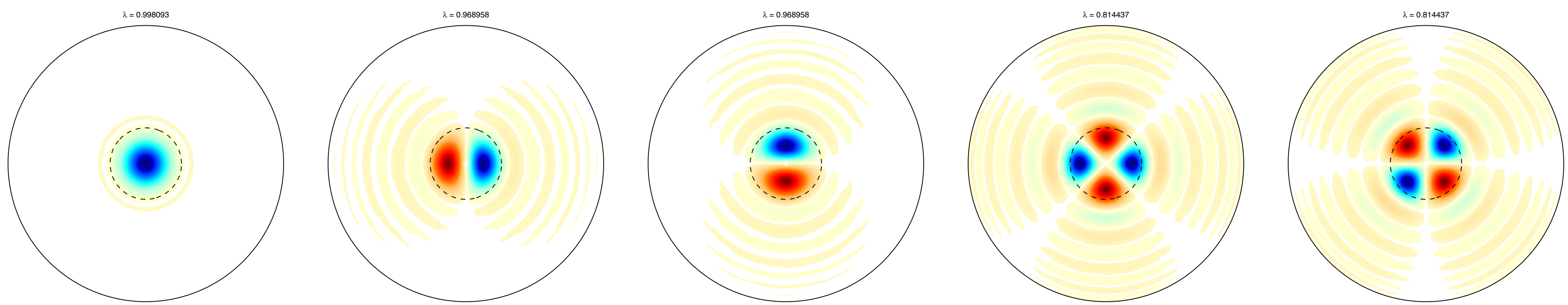

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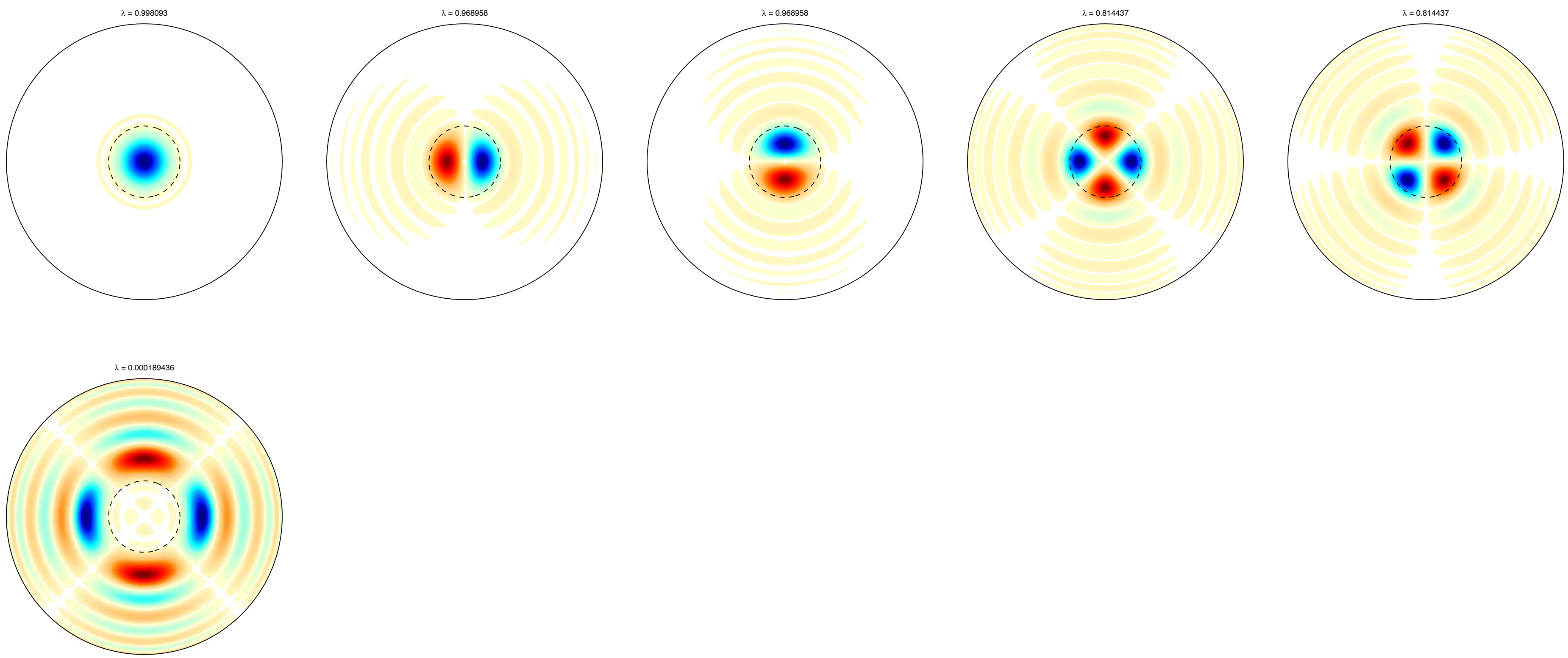

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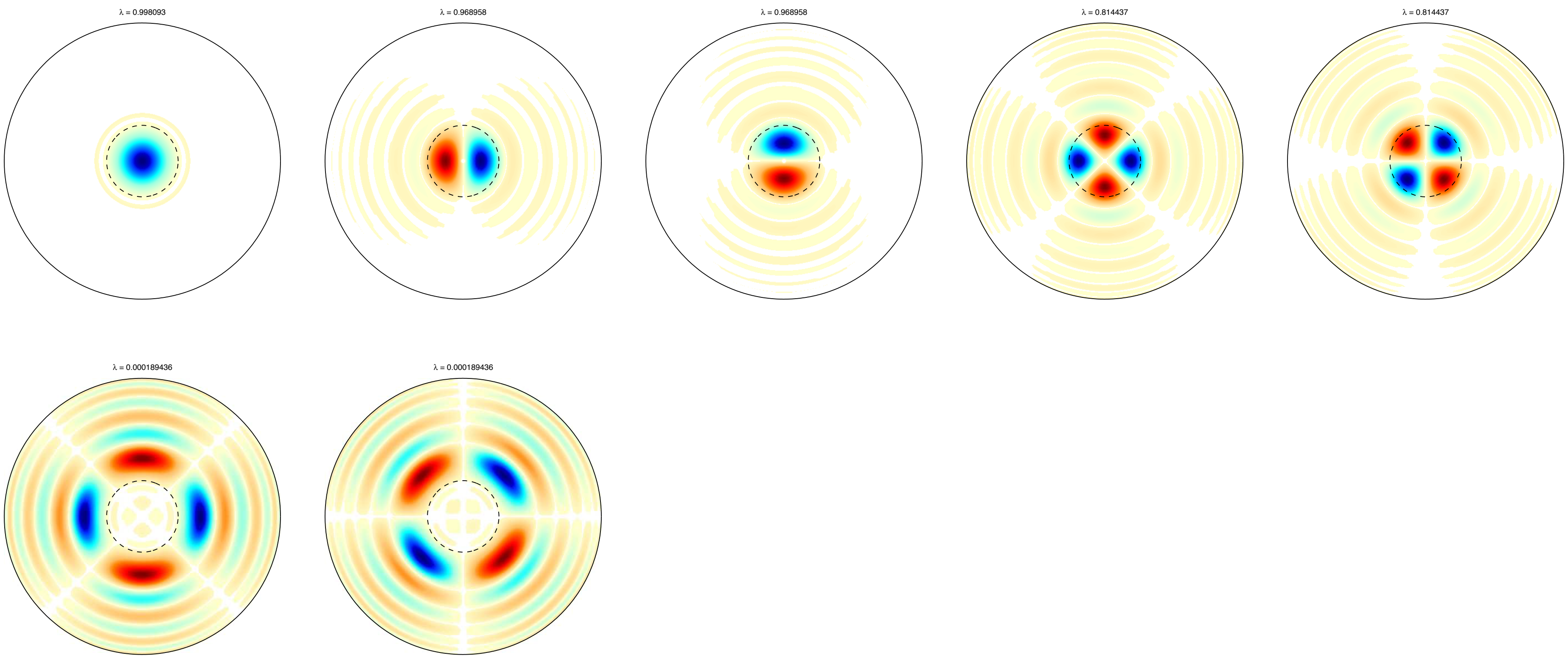

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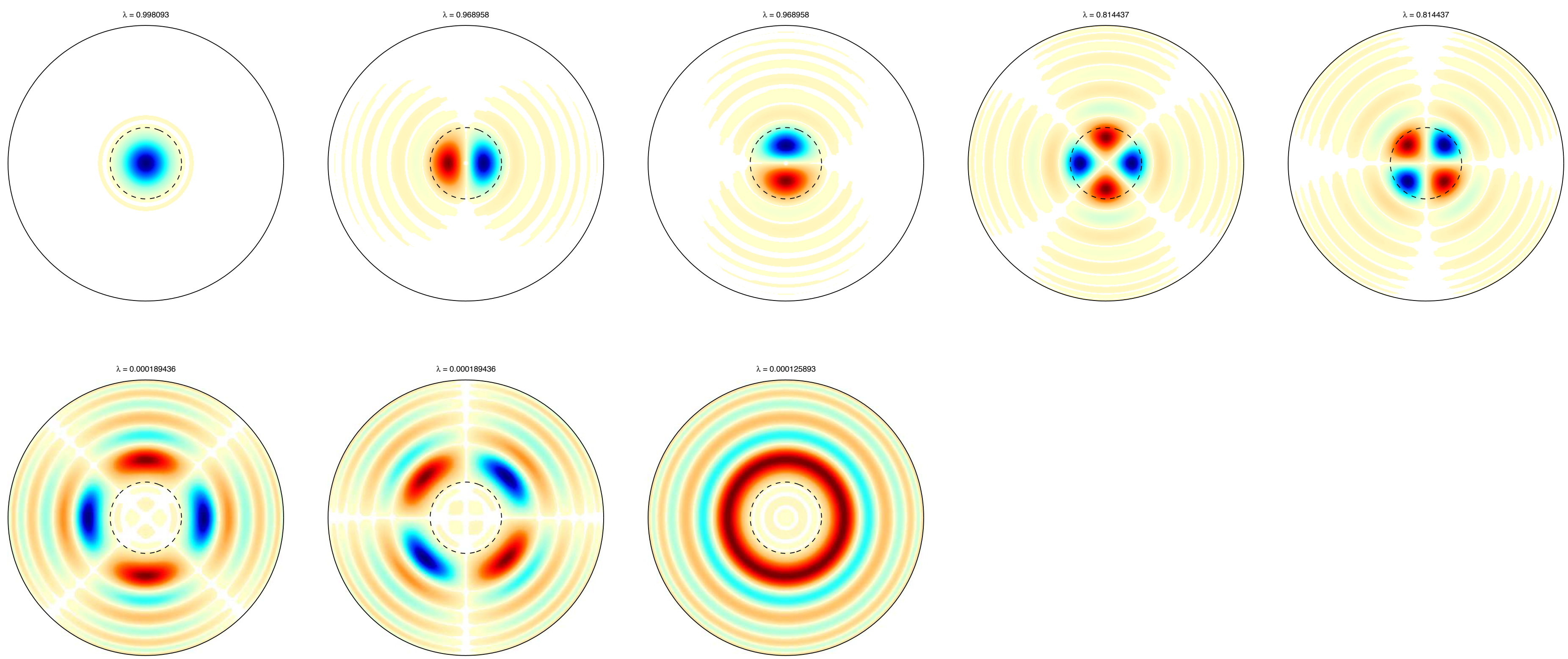

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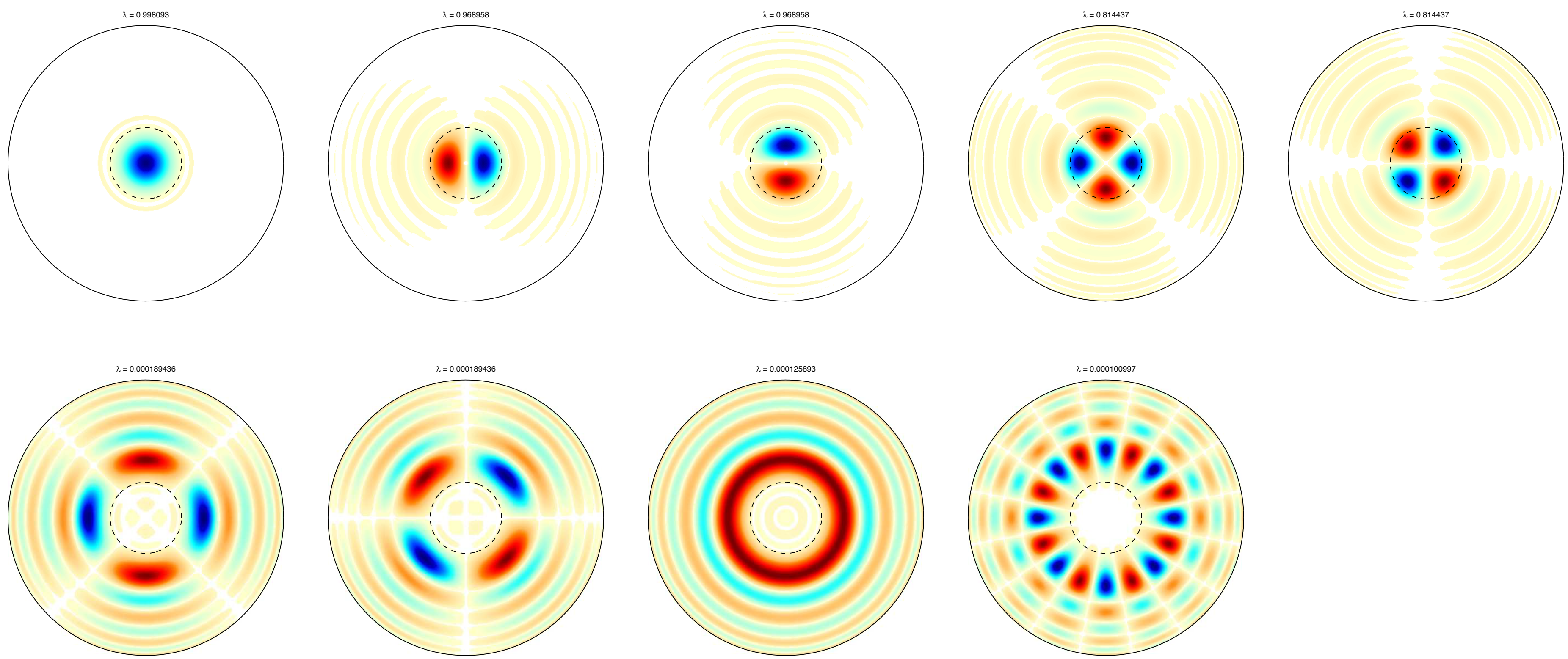

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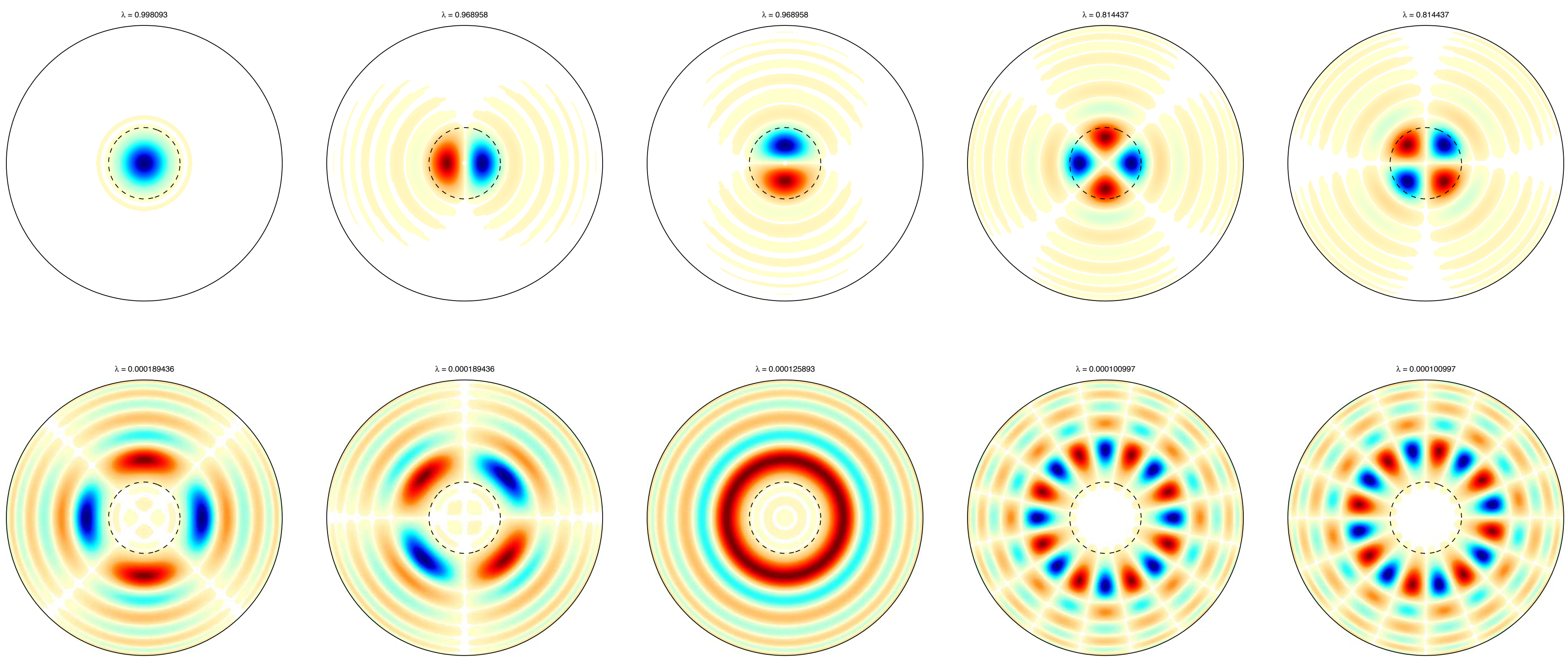

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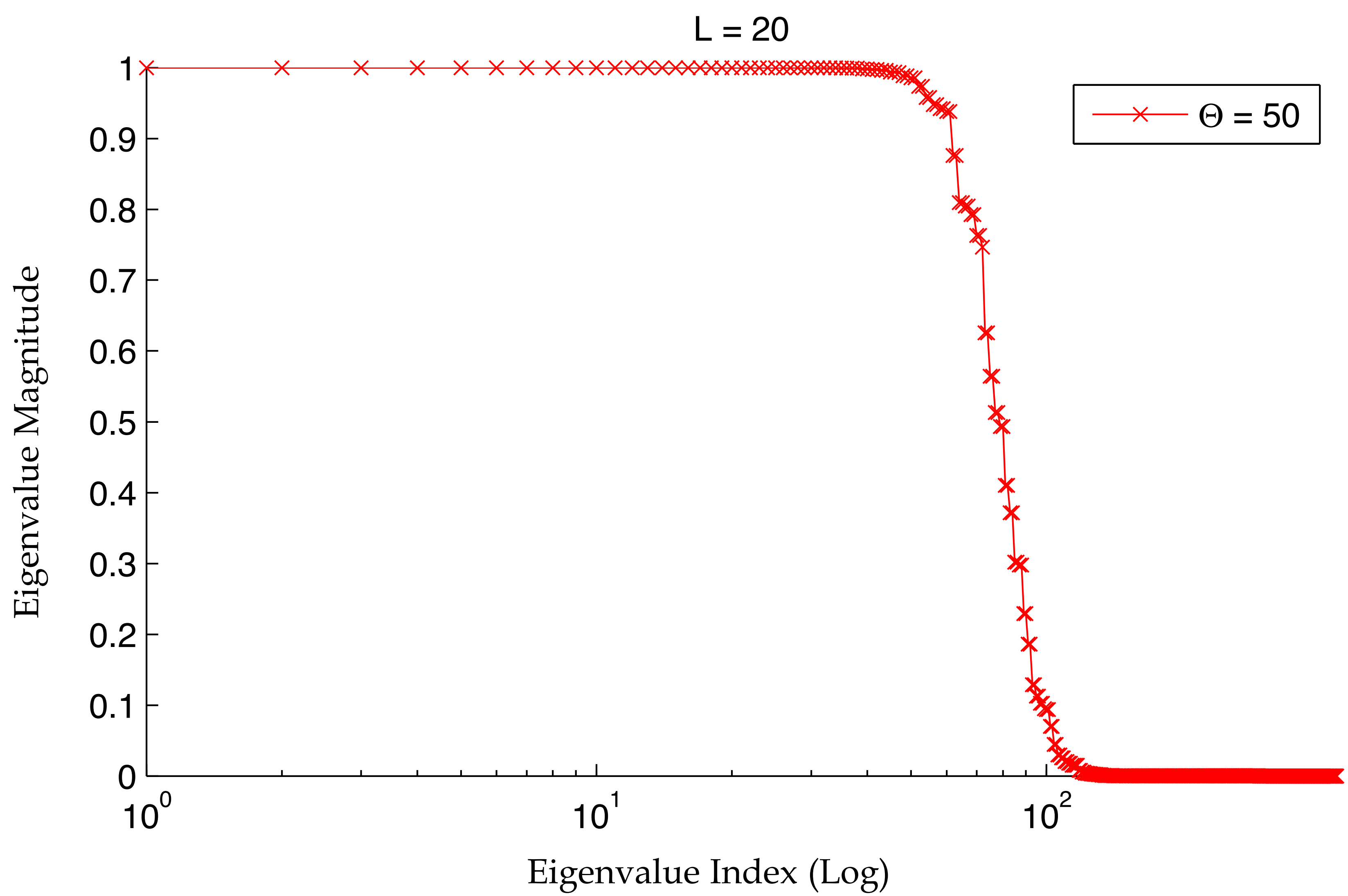

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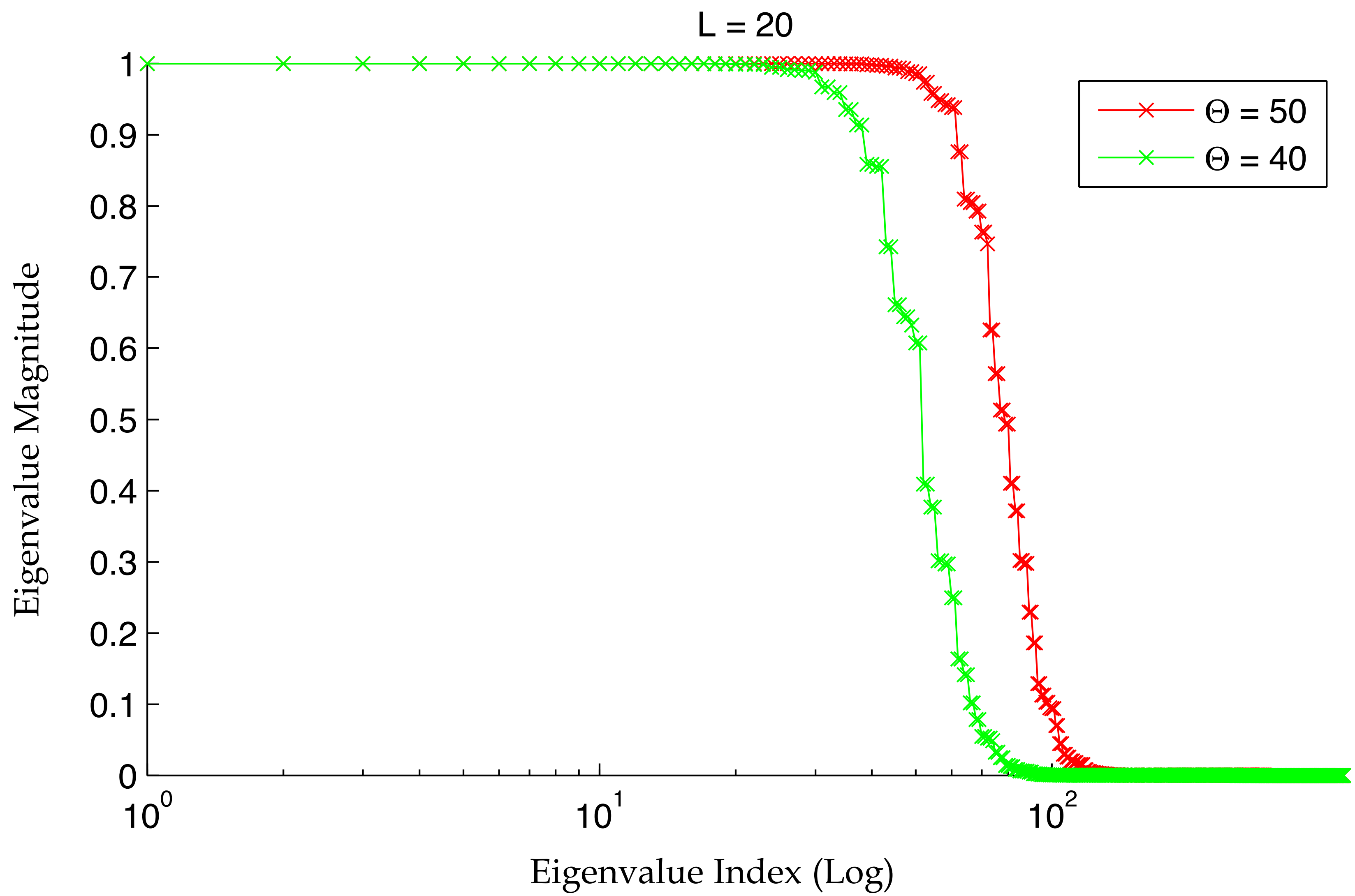

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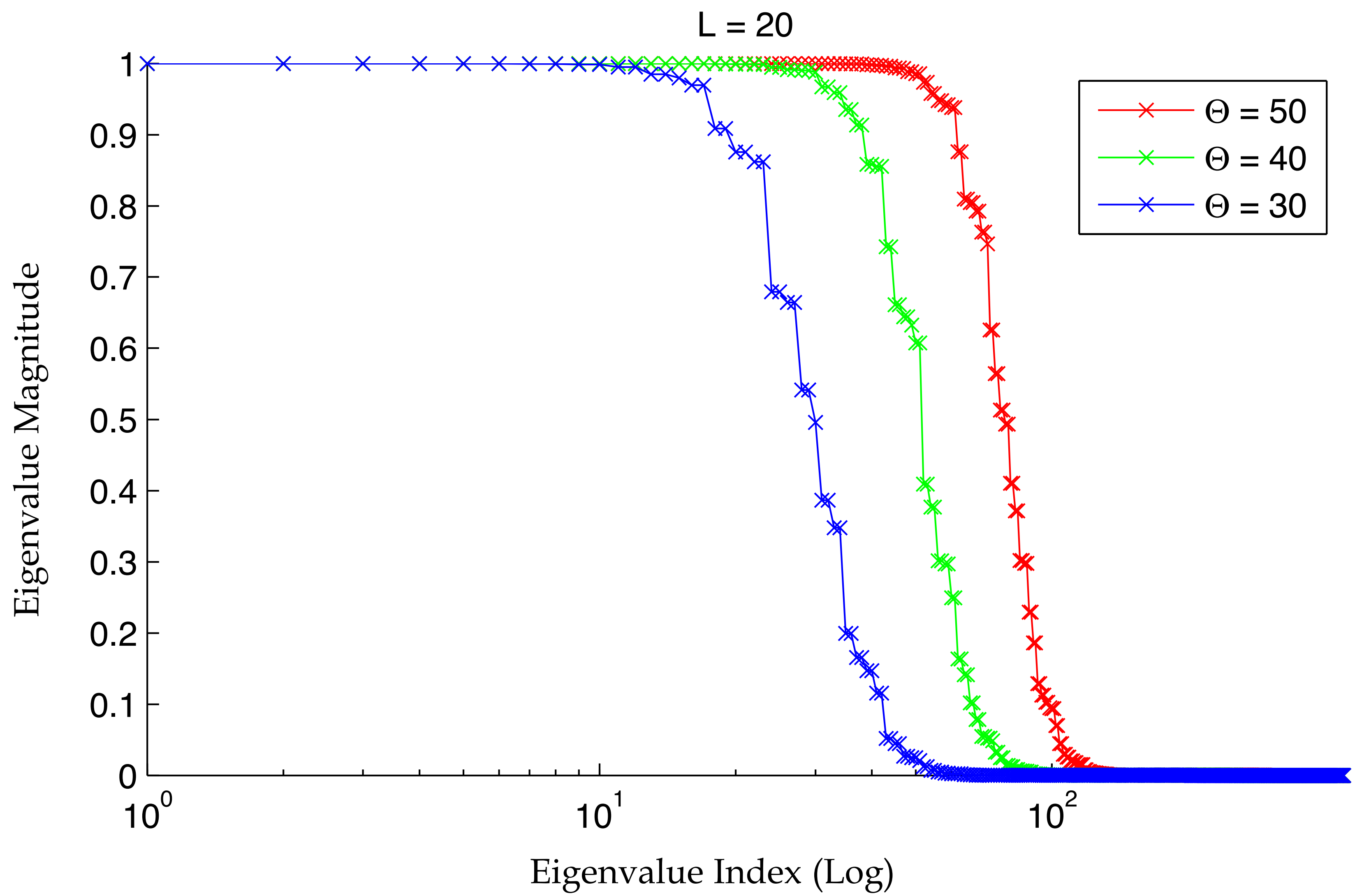

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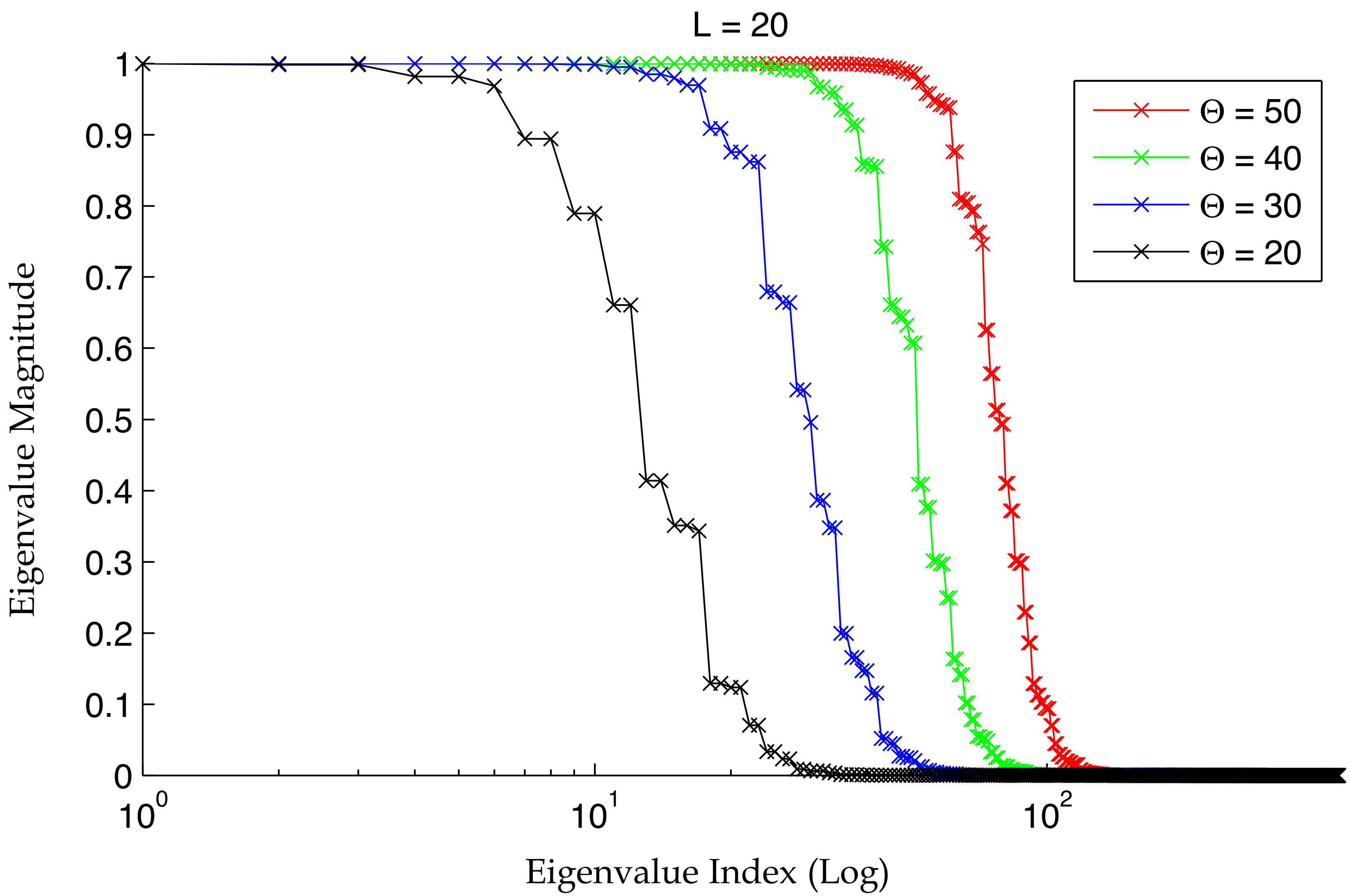

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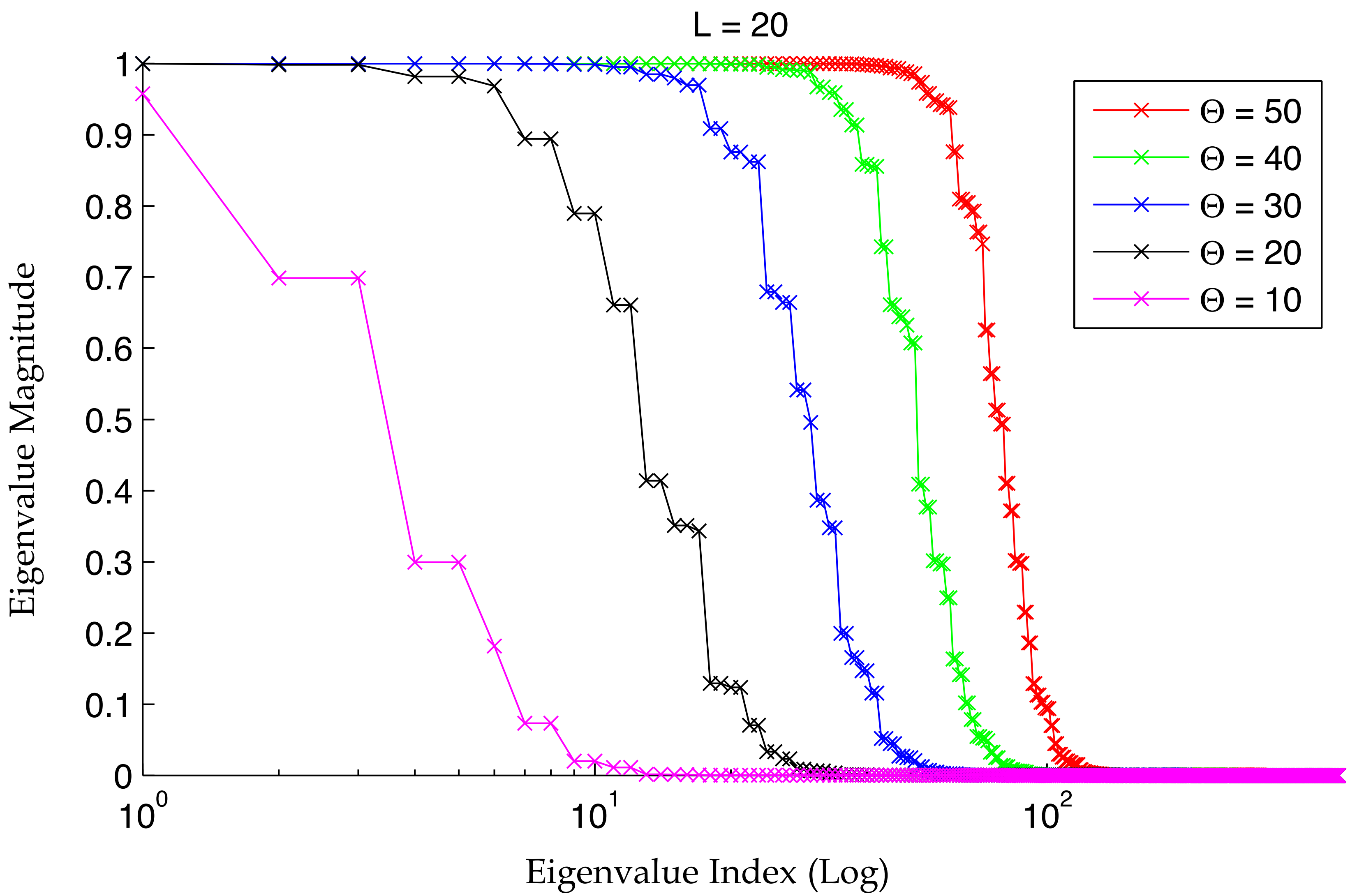

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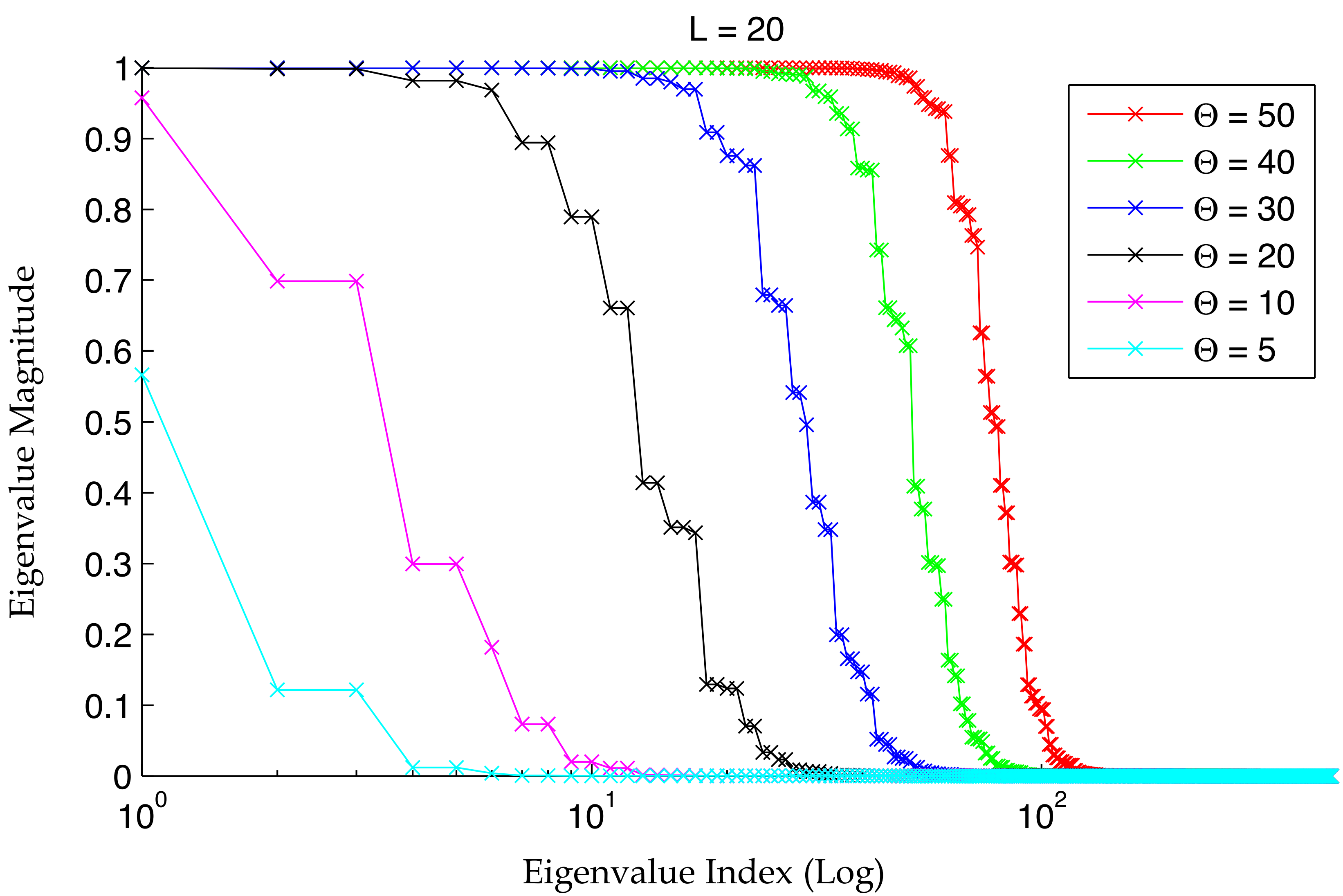

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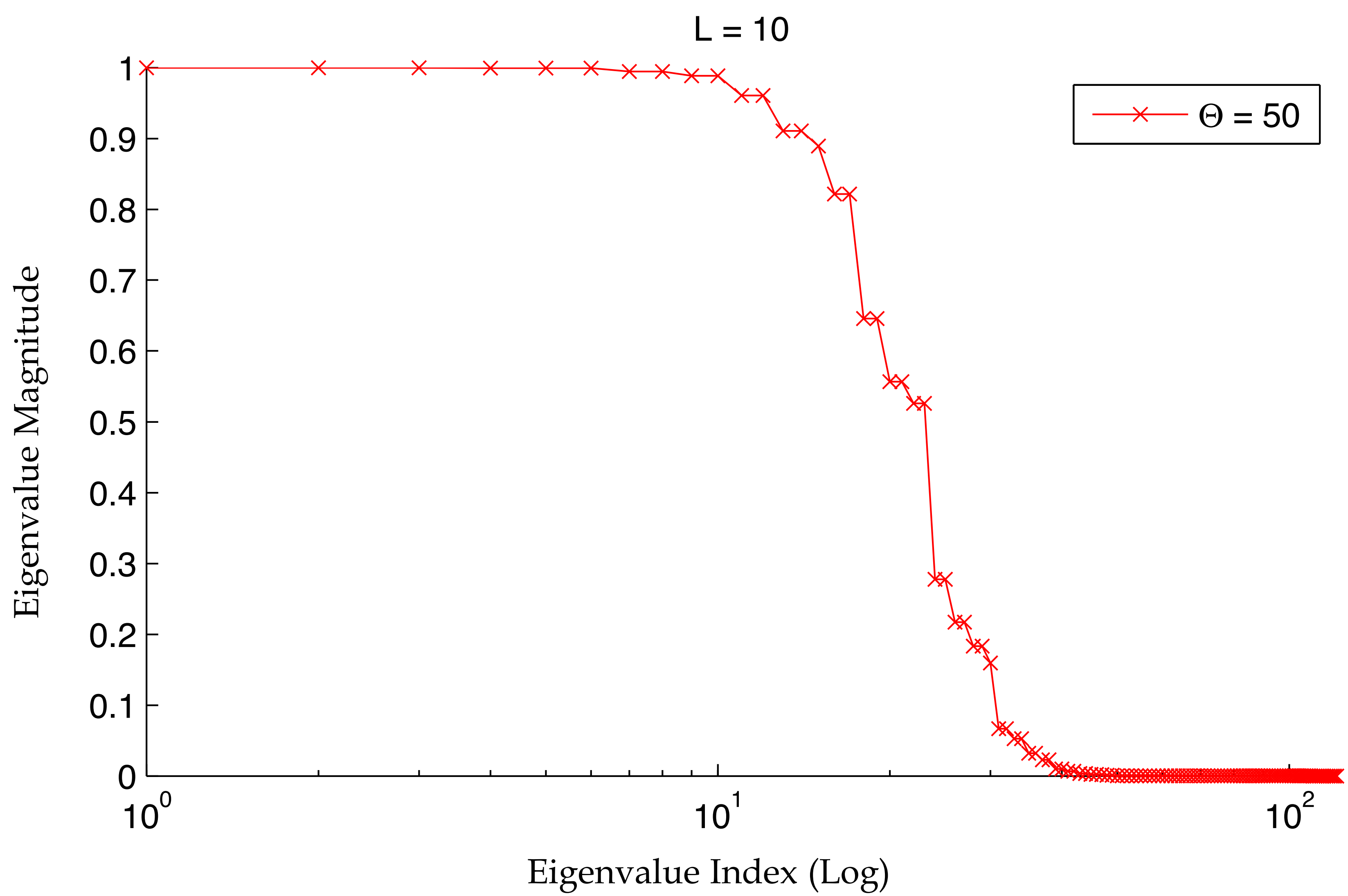

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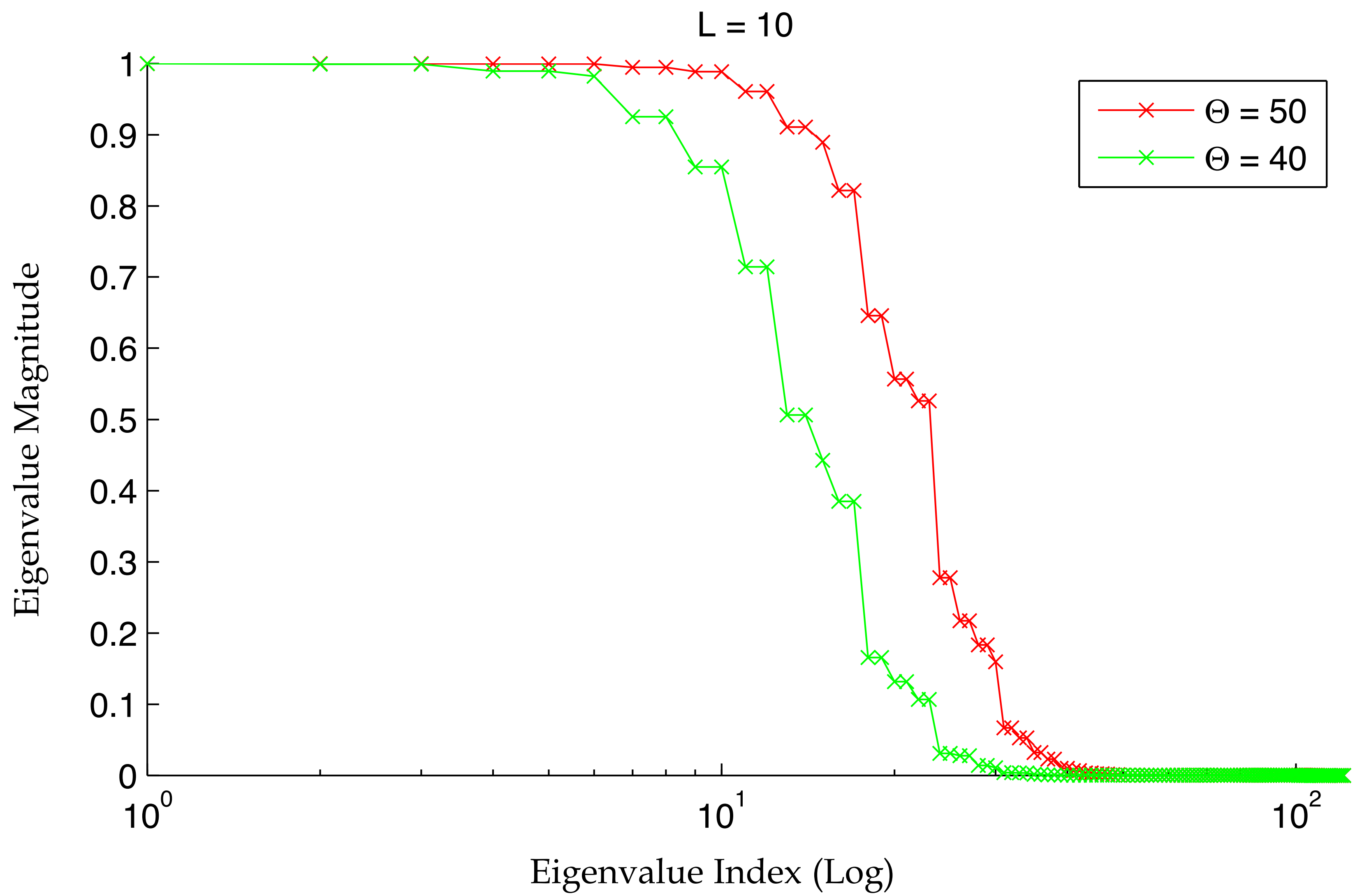

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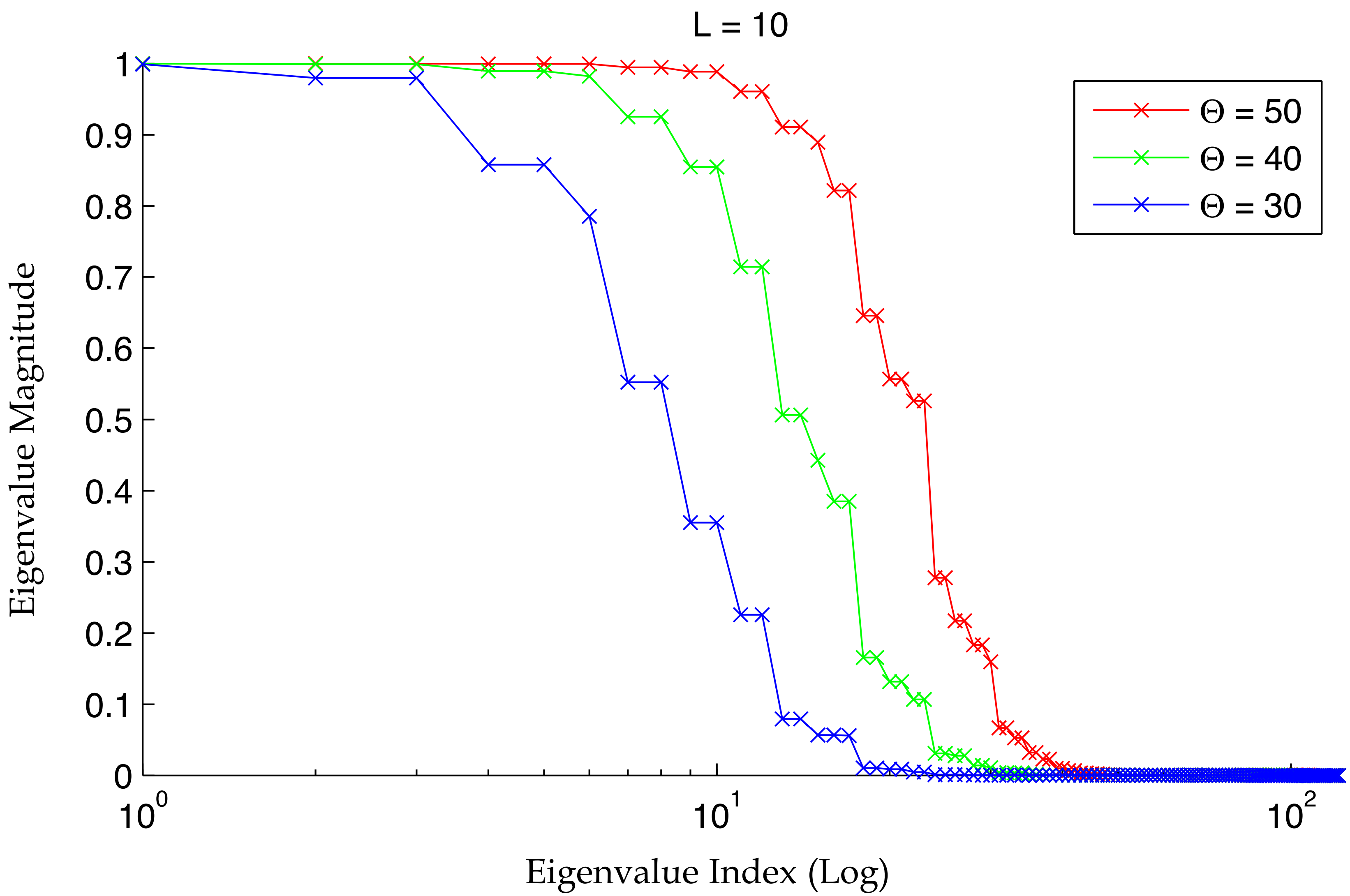

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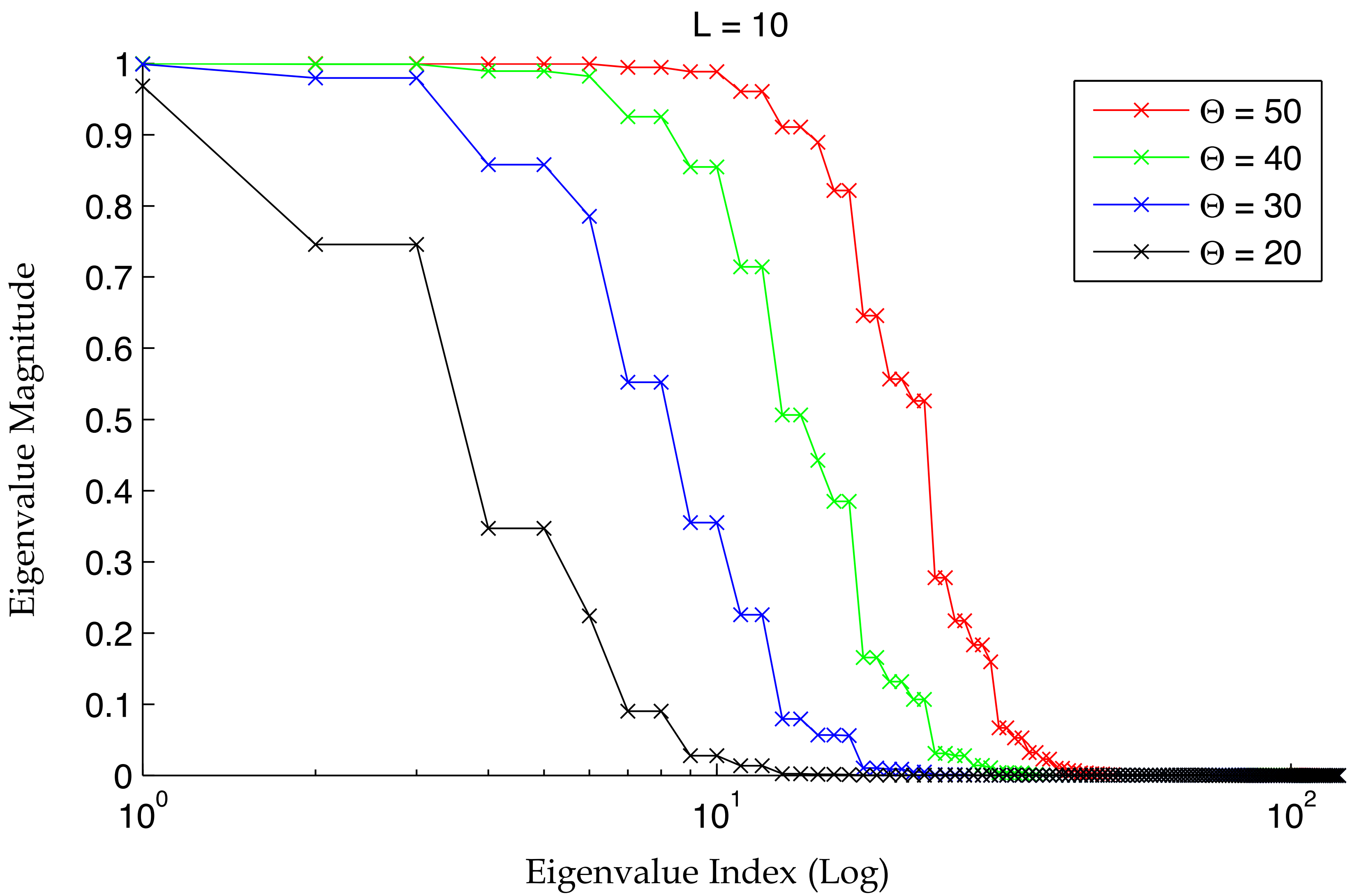

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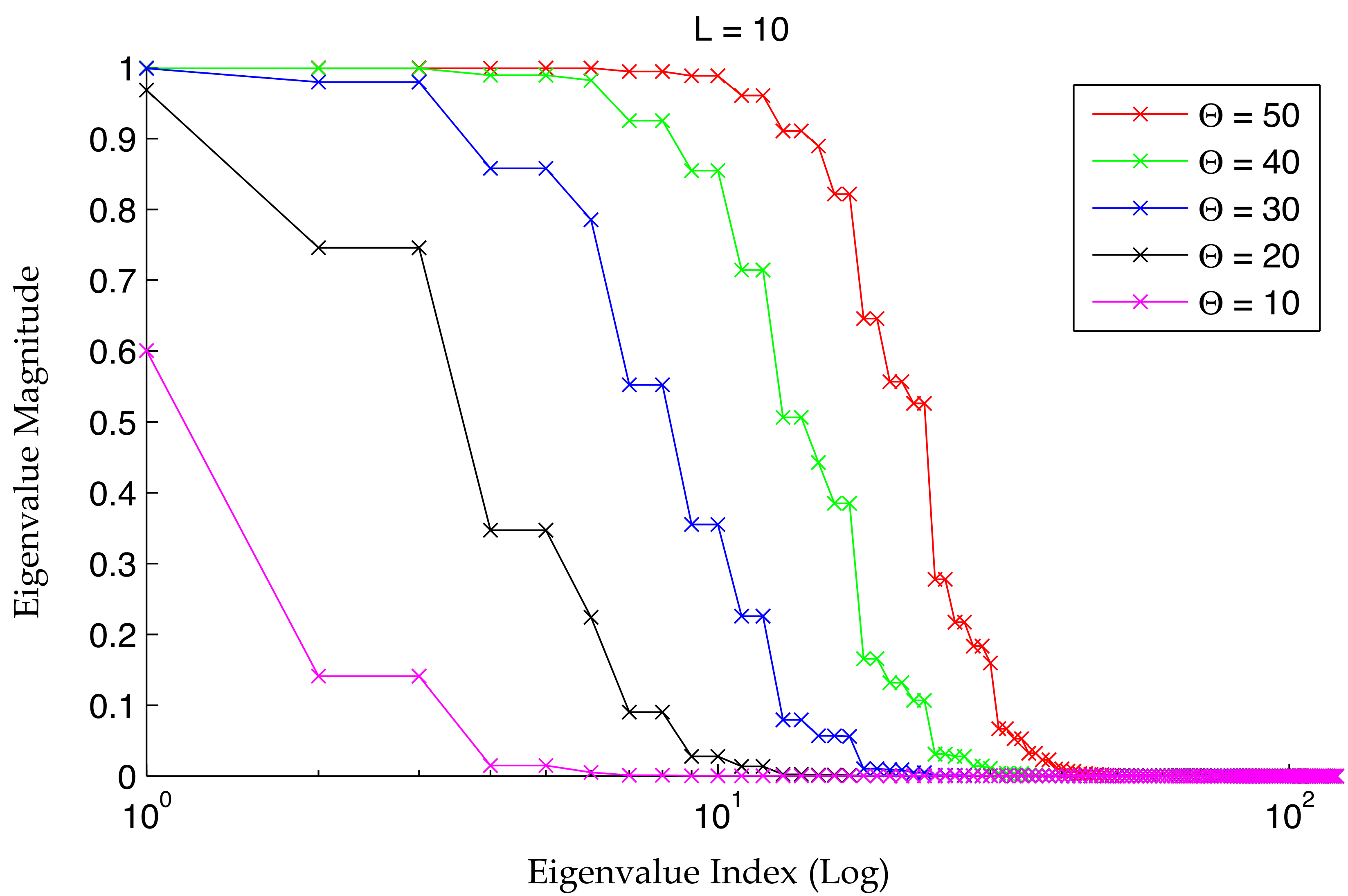

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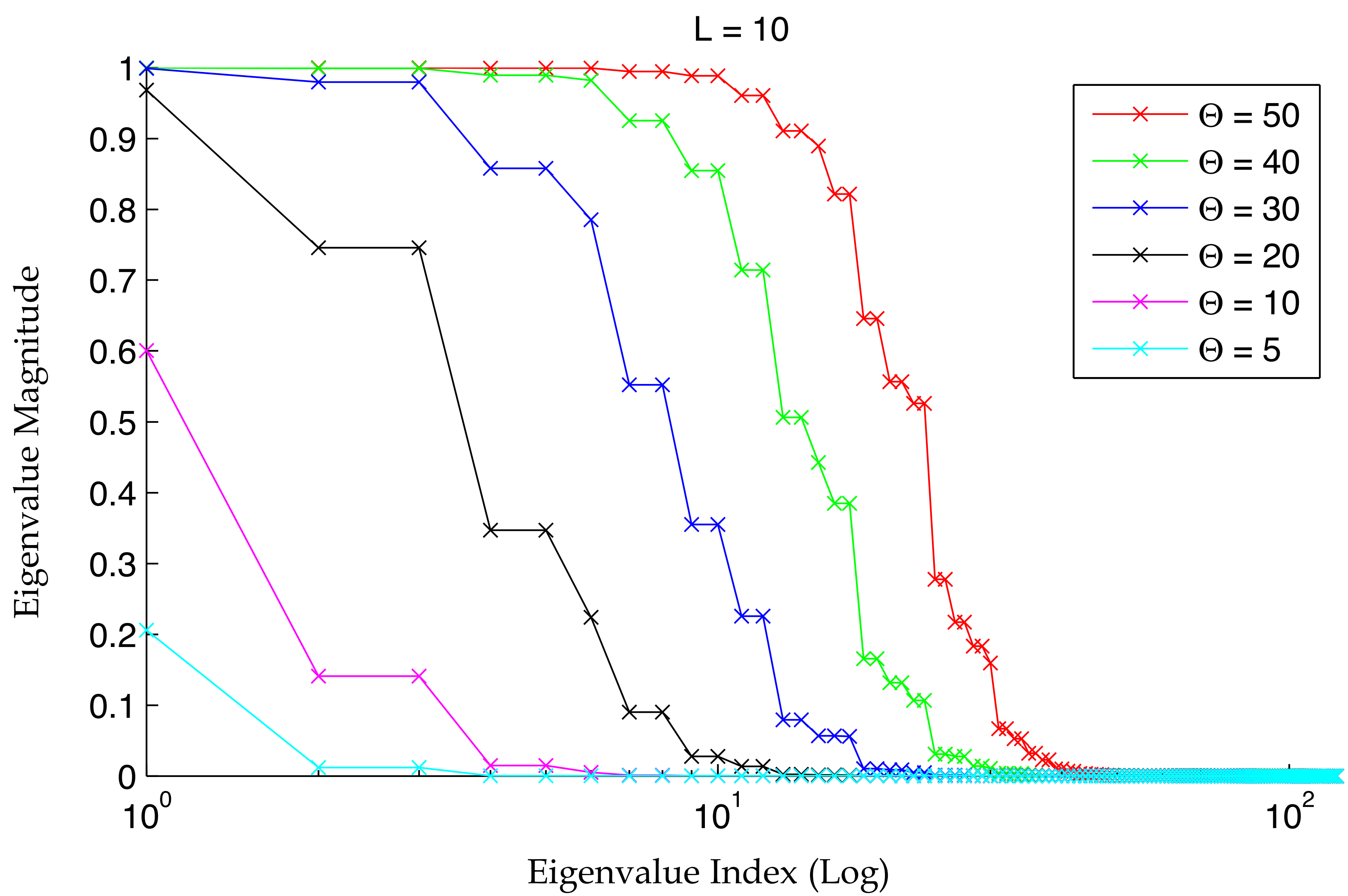

# Spatio-Spectral Concentration Theory

Spectrum (conjecture):

$$K(\epsilon) = \frac{C}{4\pi} + \left( \log \left( \frac{1-\epsilon}{\epsilon} \right) \mathcal{B}(\partial \mathcal{U}) \right) \log(C) + o(\log C)$$

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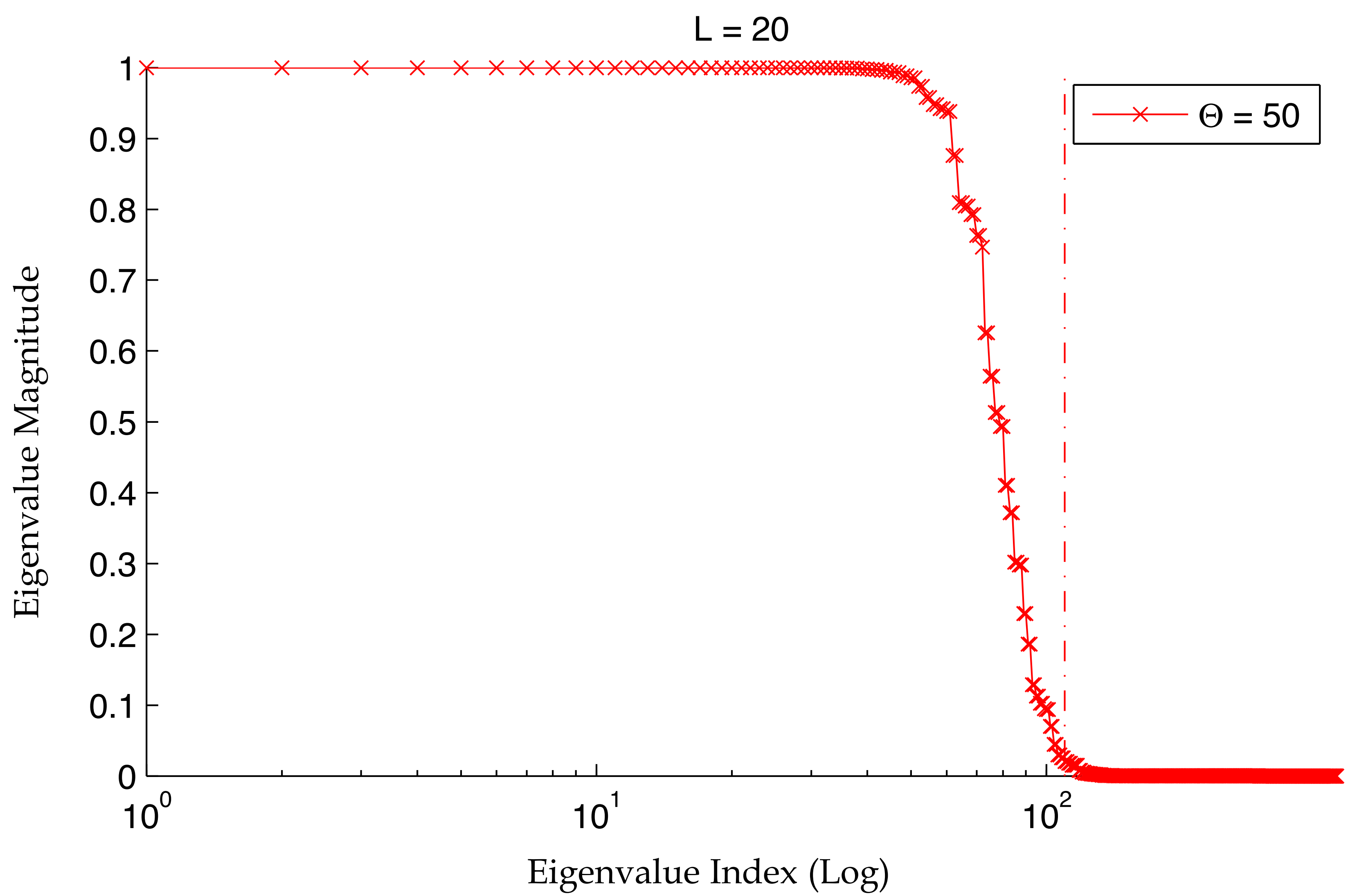

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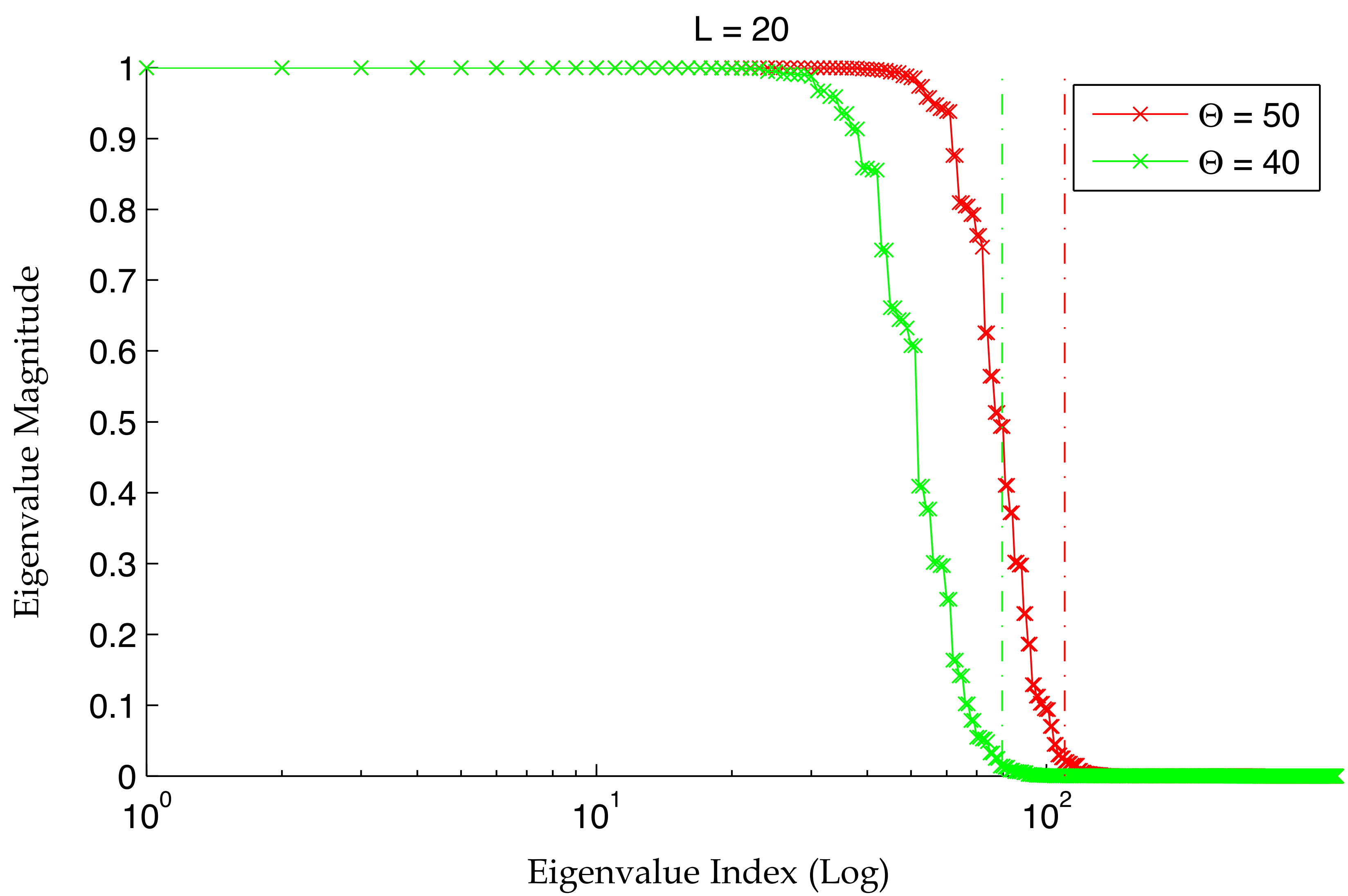

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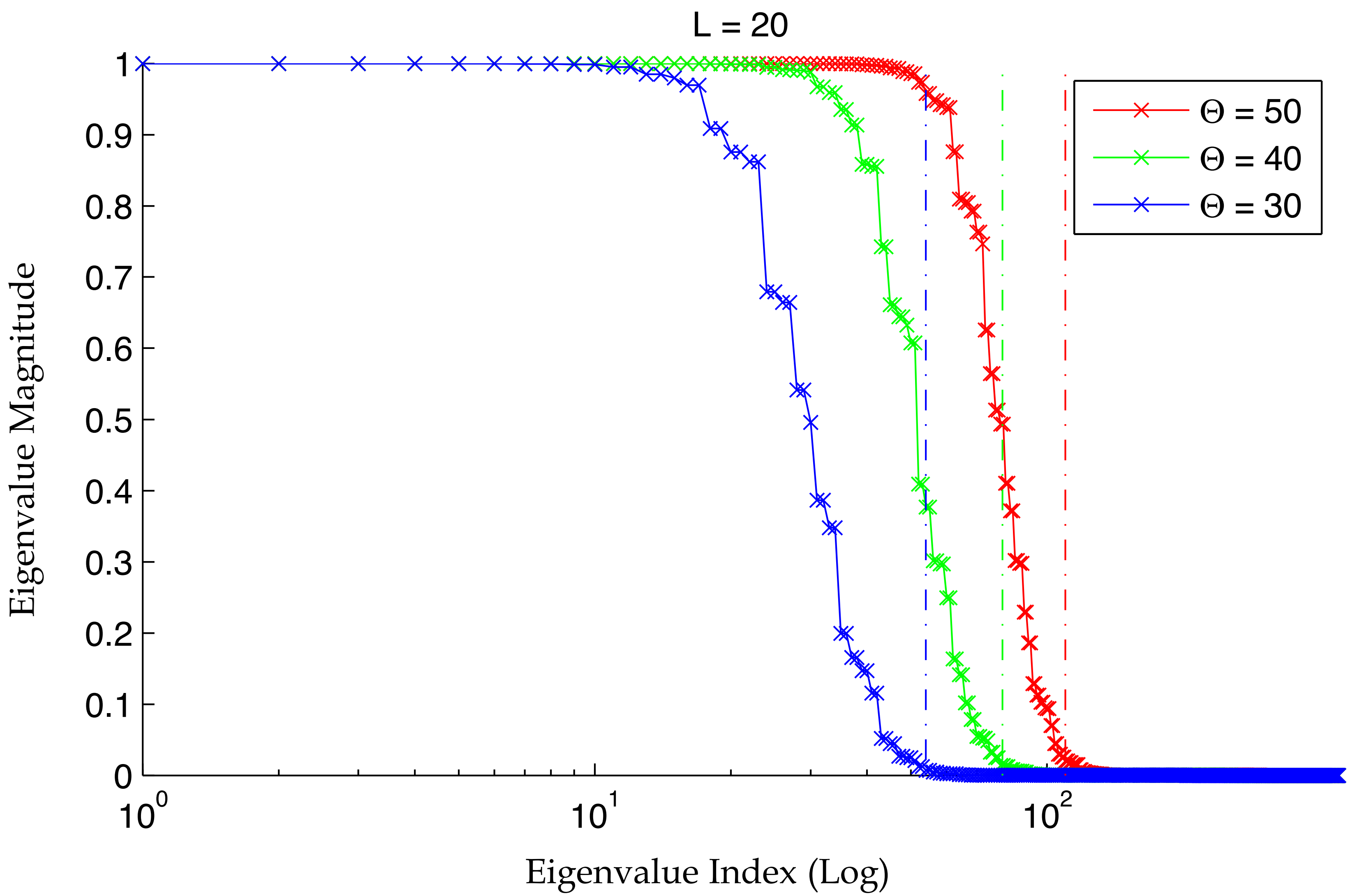

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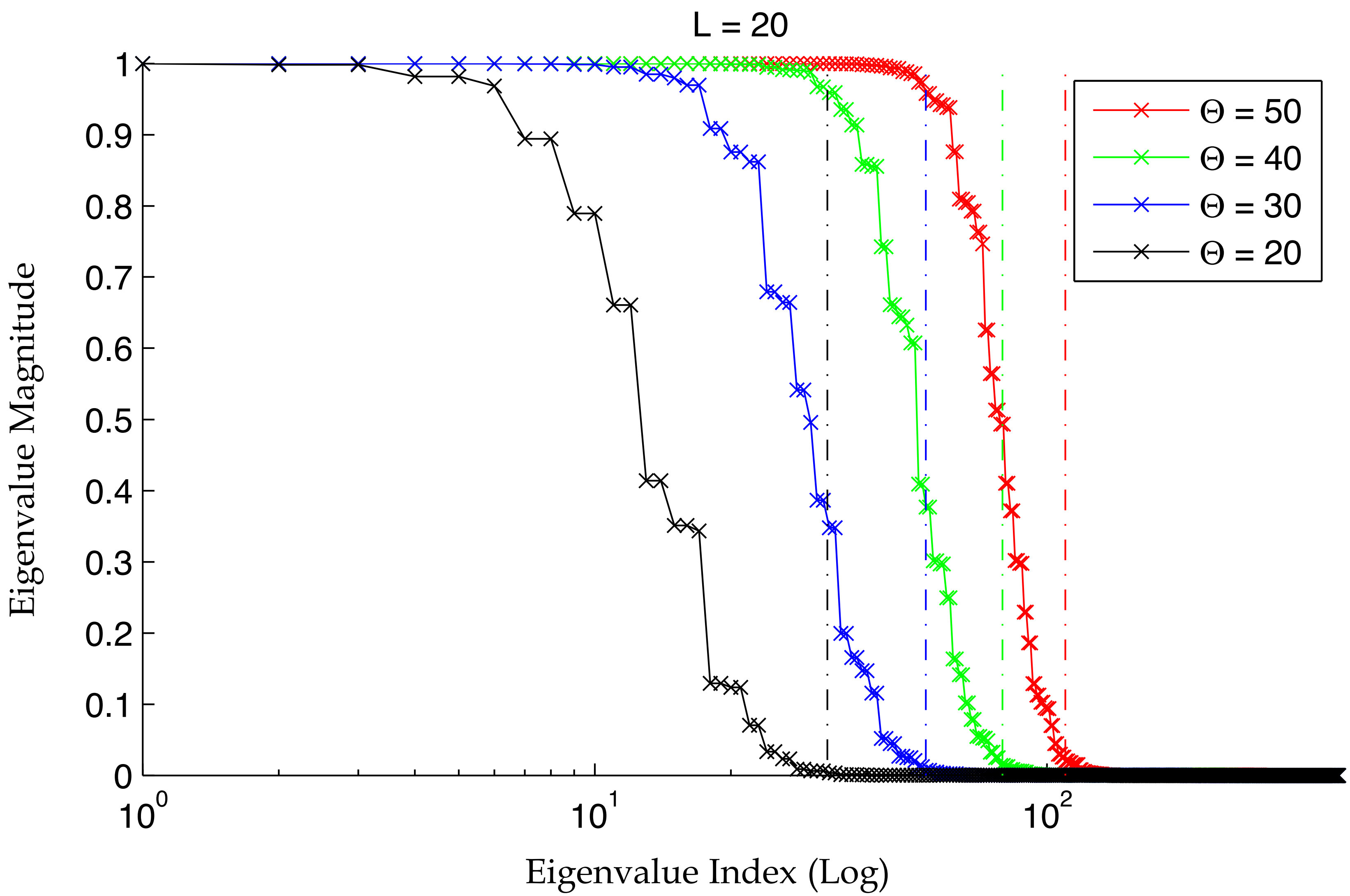

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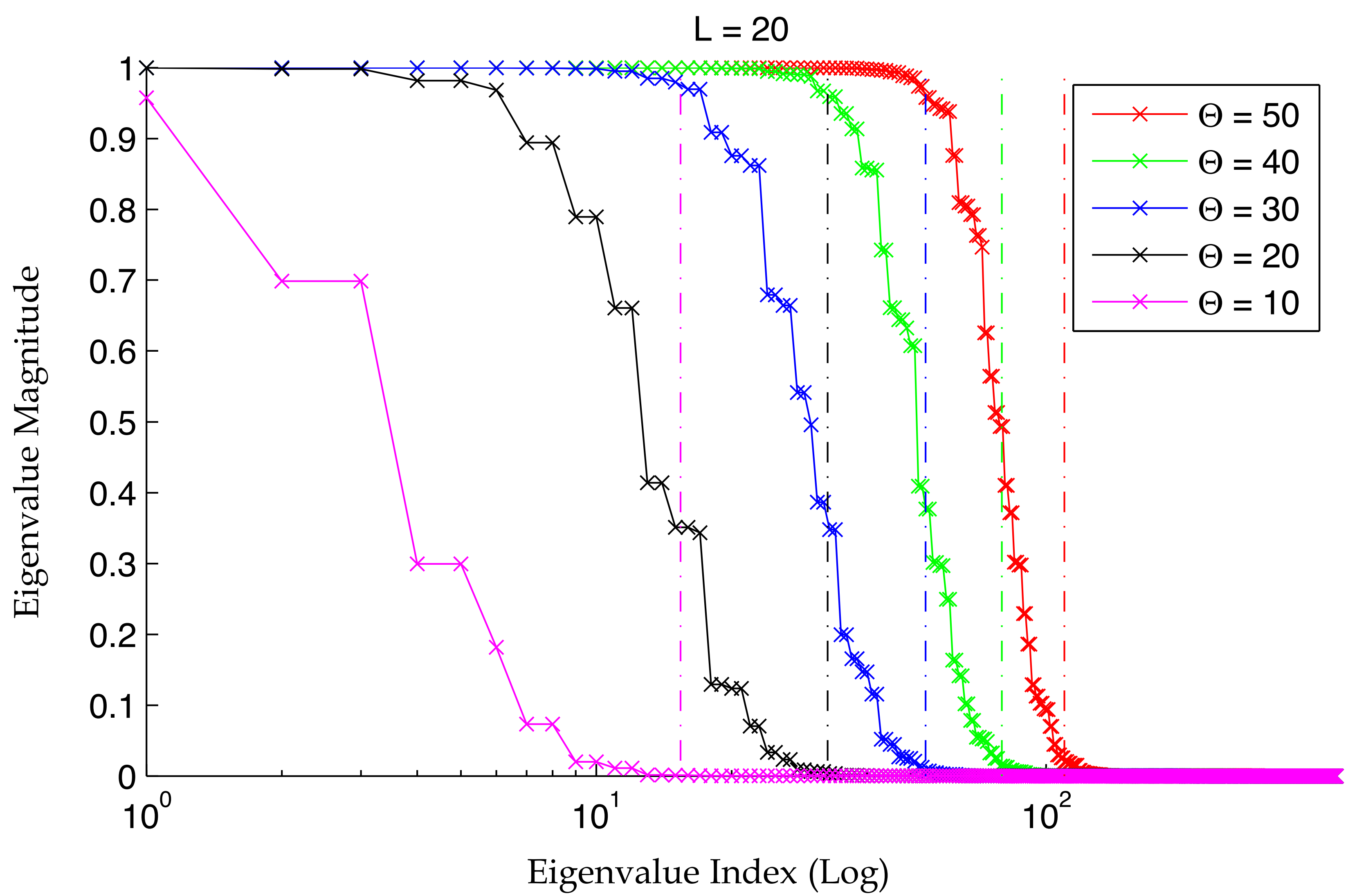

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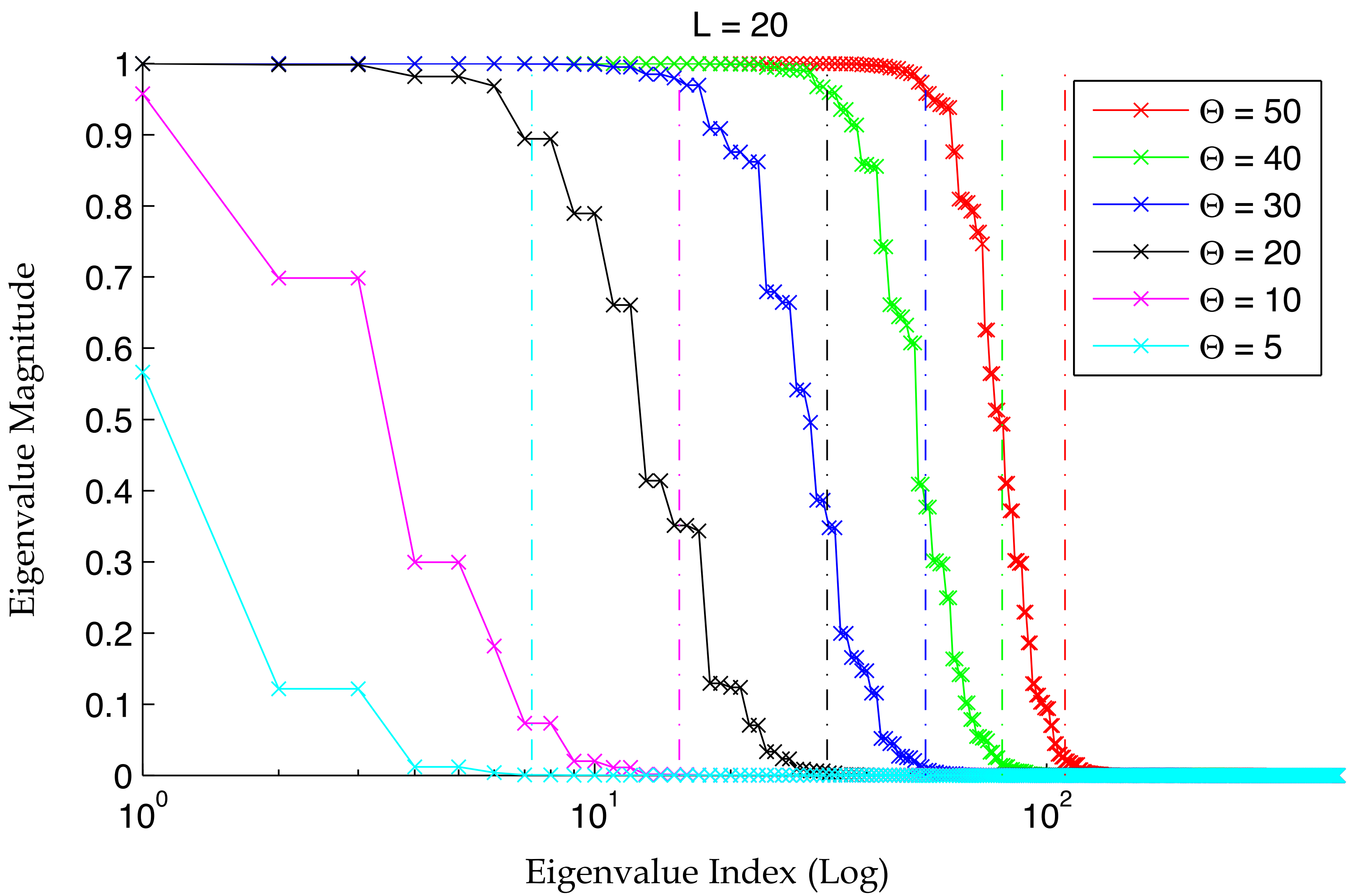

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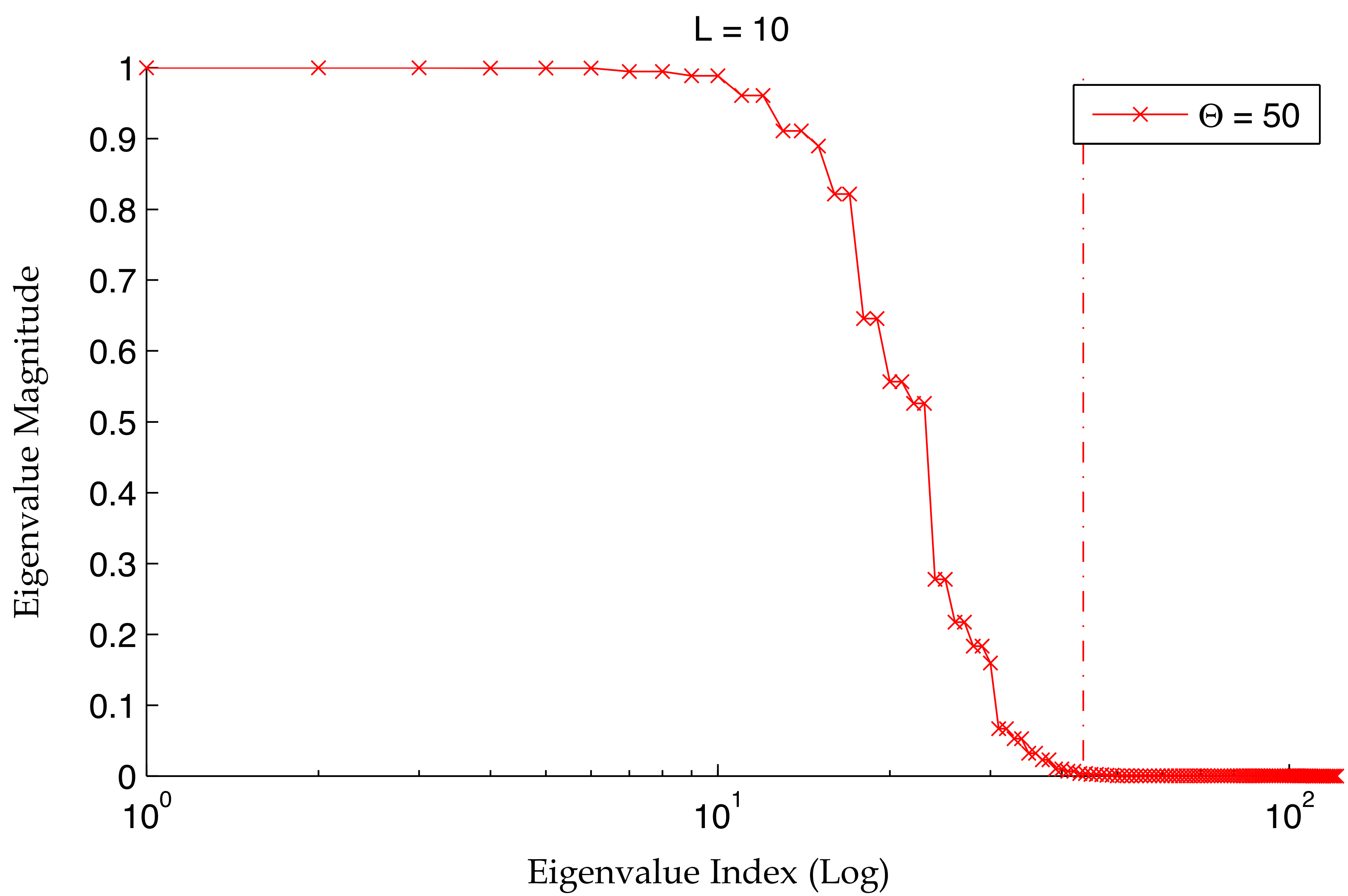

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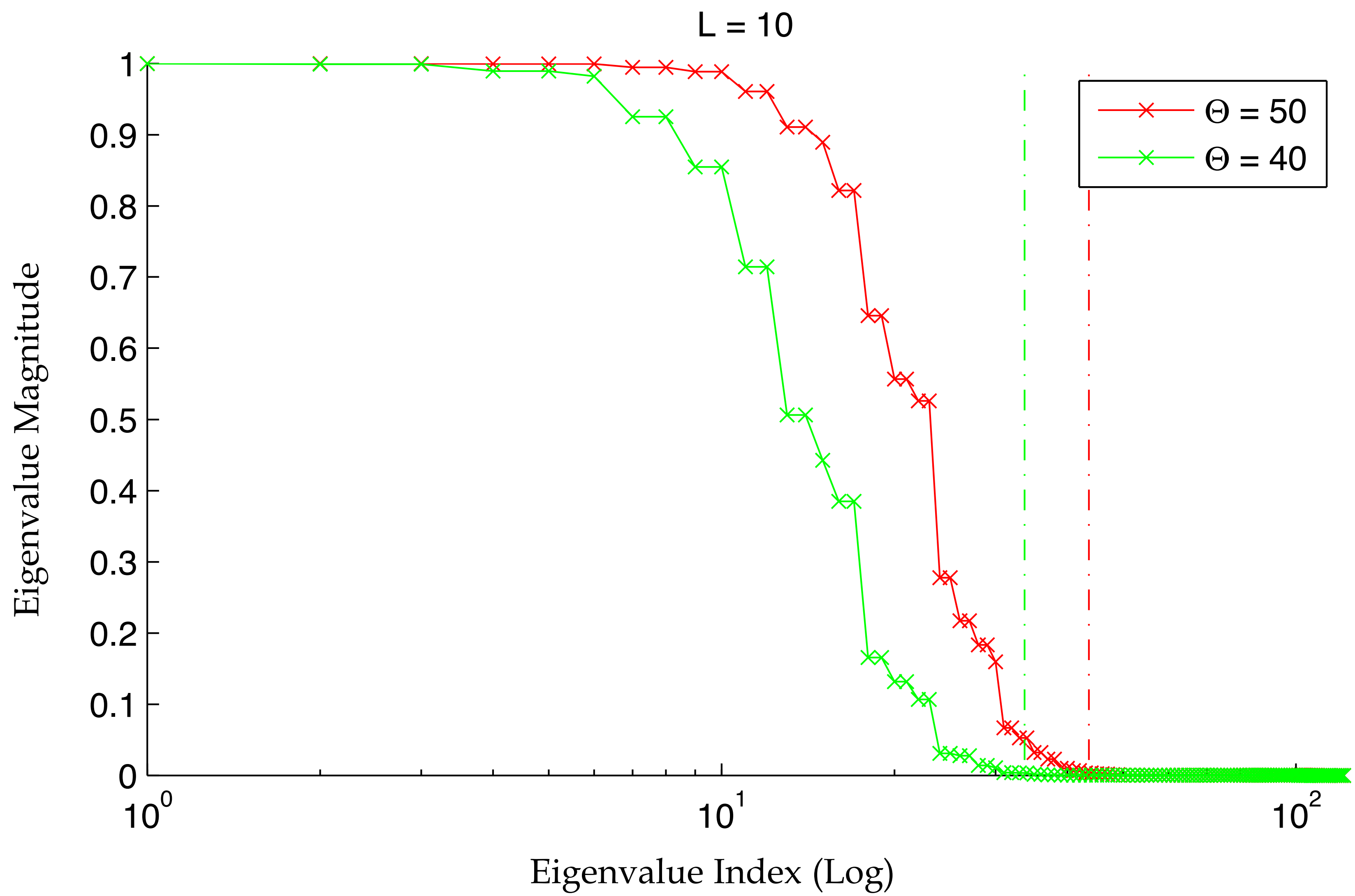

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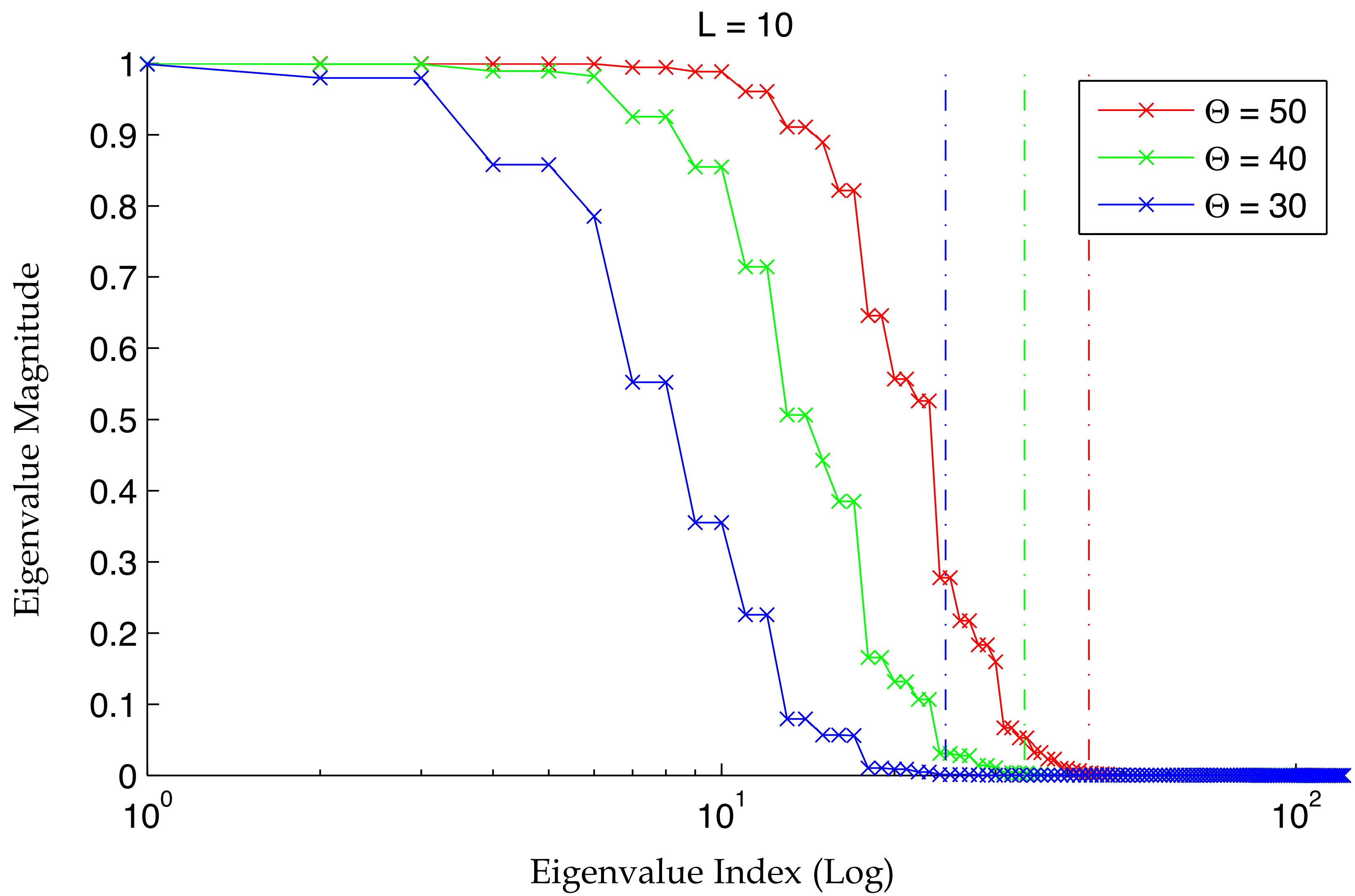

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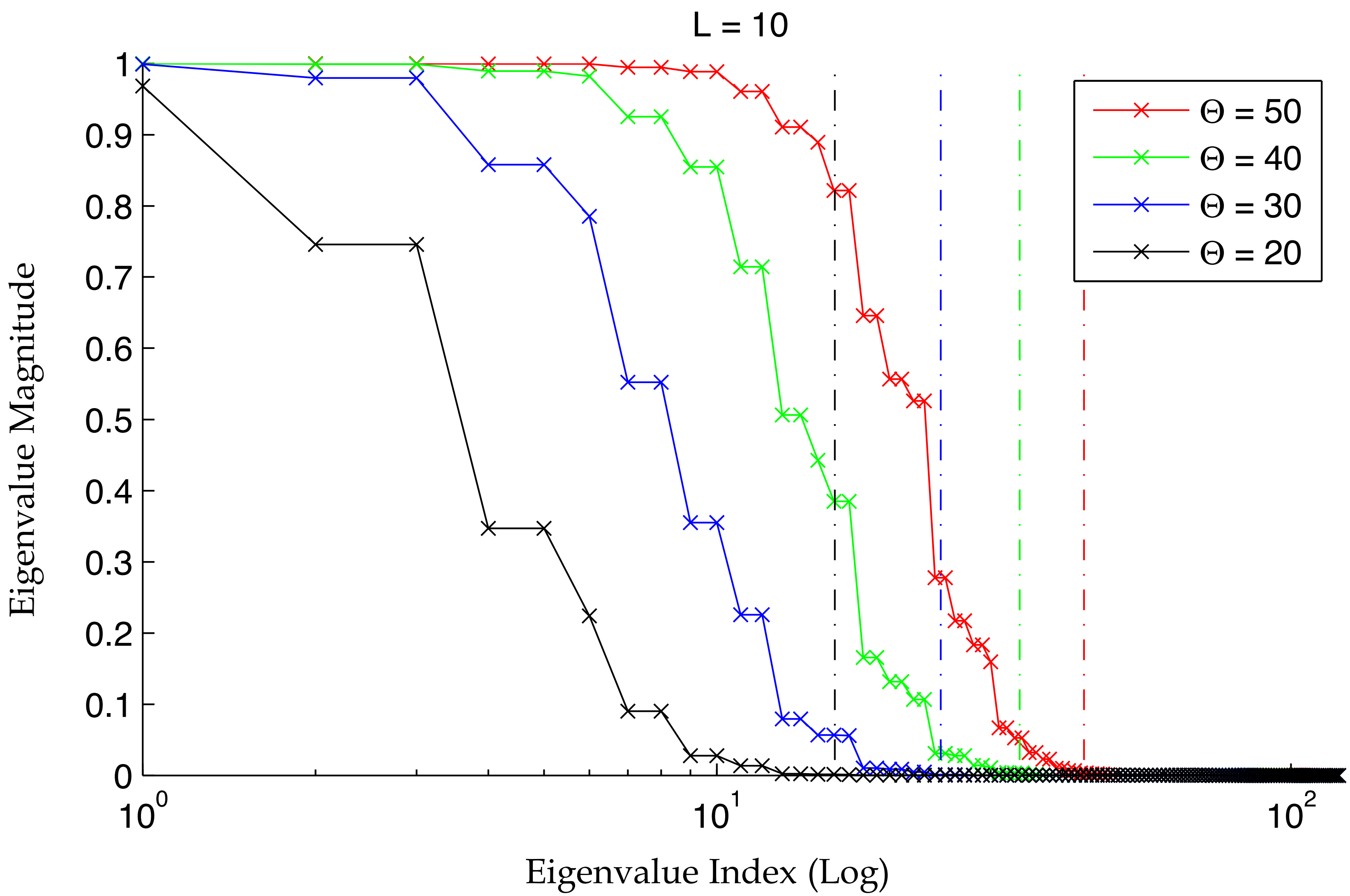

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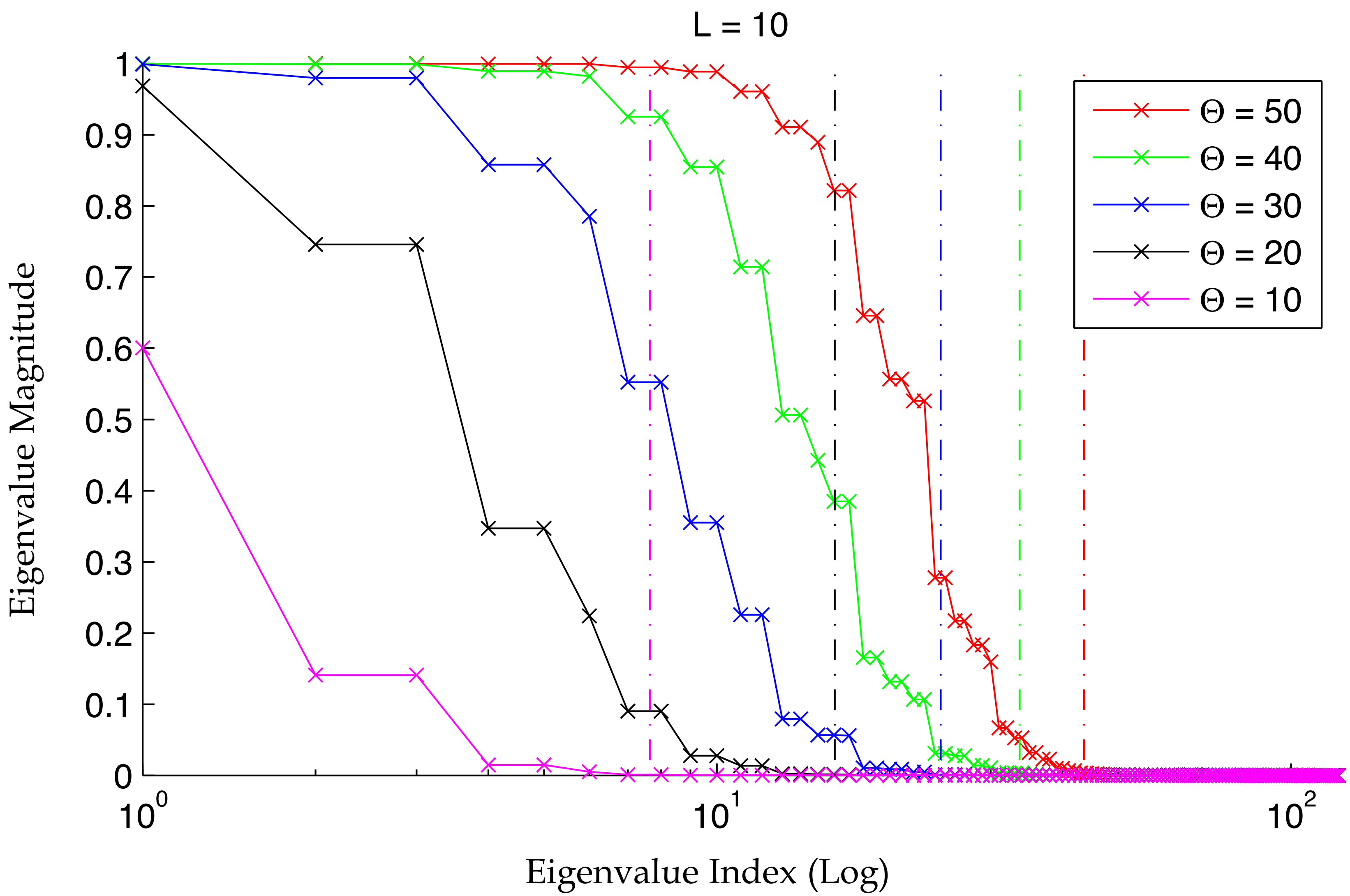

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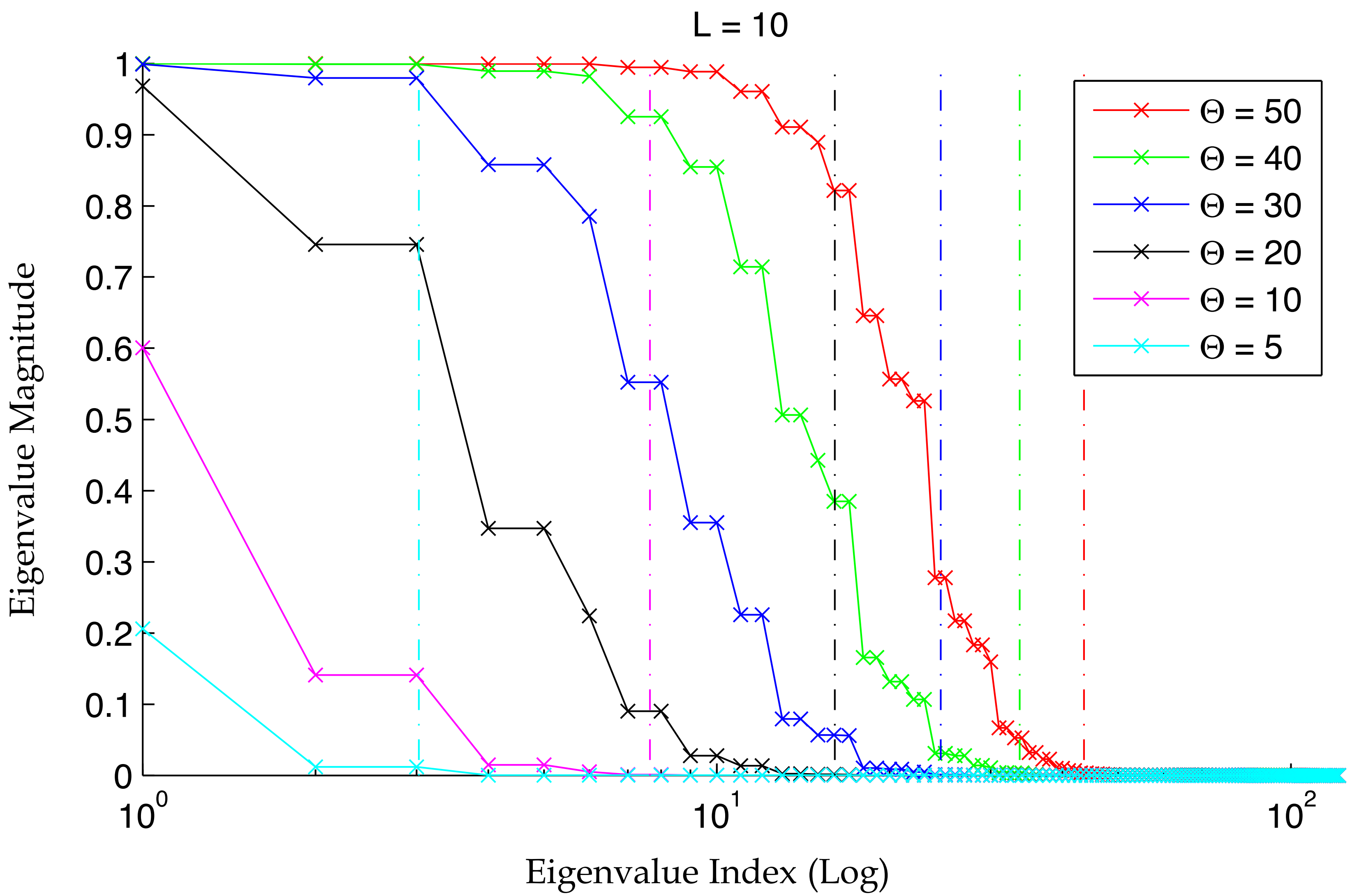

# Effective Dimension

Error is minimized if

$$B(\omega) \approx \tilde{B}(\omega) = \sum_{i=1}^{K} \tilde{b}_i \, \varphi_i \quad , \quad K \ll N$$

is approximated with $\varphi_i(\omega)$ for which

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⇒ *Optimal basis functions are Slepian functions.*

⇒ *Effective dimension is given by $K(\epsilon)$.*

# Effective Dimension

$$K(\epsilon) = \frac{C}{4\pi} + \left( \log \left( \frac{1-\epsilon}{\epsilon} \right) \mathcal{B}(\partial \mathcal{U}) \right) \log(C) .$$

# Applications

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  - Optimal cluster size.
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- Exploitation of coherence of light transport in sampling-based algorithms.

# Future Work

- Boundary function for $K(\epsilon)$.

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# Conclusion

- Characterization of effective dimension of light transport in a local neighborhood.
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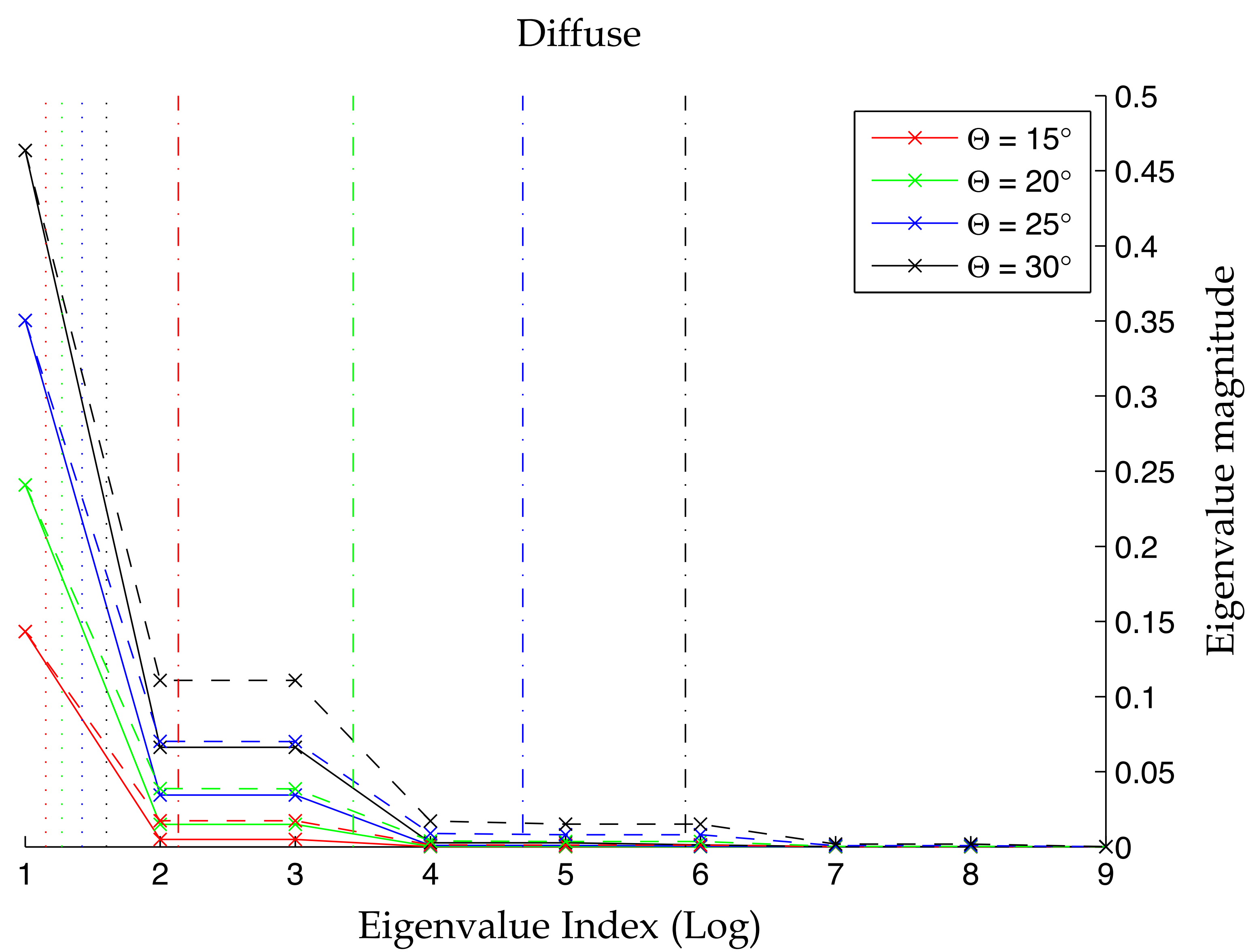
- Characterization of effective dimension of light transport in a local neighborhood.
  - Improved estimate for dimensionality.
  - Closed form expression for basis functions.
- Introduction of Slepian functions.
  - Efficient representation of localized signals.
  - Advantages of Spherical Harmonics.

More information and source code:

www.dgp.toronto.edu/people/lessig/effective-dimension/

# Effective Dimension



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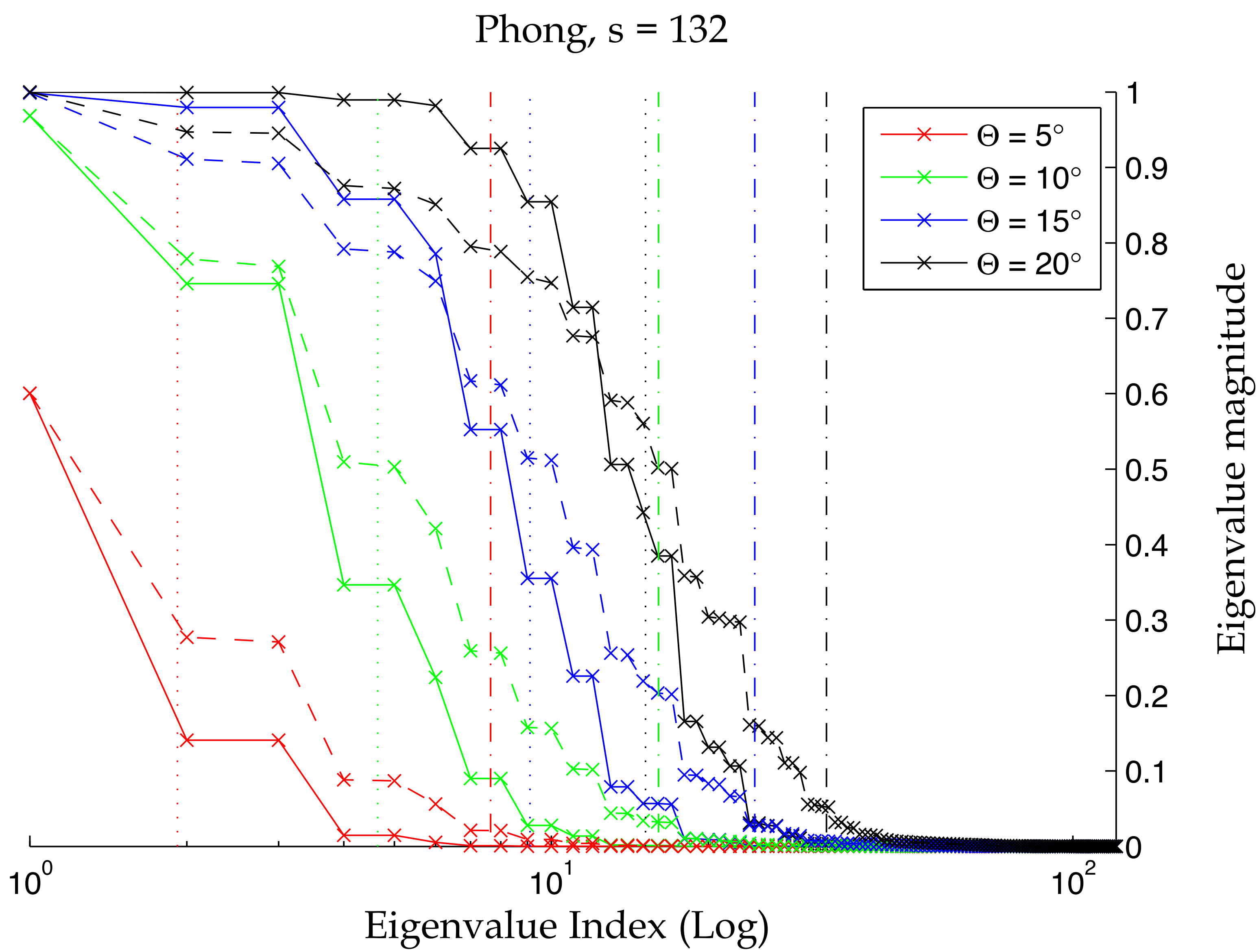

# Related Work

- Coherence has been exploited in precomputed radiance transfer for some time.[1]
- Mahajan et al.[2] studied the effective dimension in flatland and discussed extension to 3D.
- Shading equation has been studied previously by Ramamoorthi and co-workers[3] using assumptions similar to ours.

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[1] Liu X, Sloan P, Shum H, Snyder J. *All-Frequency Precomputed Radiance Transfer for Glossy Objects*. In: Eurographics Symposium on Rendering 2004.; 2004.; Sloan P, Hall J, Hart J, Snyder J. *Clustered Principal Components for Precomputed Radiance Transfer*. In: SIGGRAPH '03: ACM SIGGRAPH 2003 Papers. New York, NY, USA: ACM Press; 2003:382-391.

[2] Mahajan D, Shlizerman IK, Ramamoorthi R, Belhumeur P. *A Theory of Locally Low Dimensional Light Transport*. ACM Trans. Graph. 2007;26(3 (Proceedings of ACM SIGGRAPH 2007):1-9.

[3] Ramamoorthi R, Hanrahan P. *A Signal-Processing Framework for Inverse Rendering*. International Conference on Computer Graphics and Interactive Techniques. 2001. Available at: http://portal.acm.org/citation.cfm?id=383271; Ramamoorthi R, Hanrahan P. *A Signal-Processing Framework for Reflection*. ACM Transactions on Graphics (TOG). 2004;23(4).; Ramamoorthi R, Koudelka M, Belhumeur P. *A Fourier Theory for Cast Shadows*. IEEE Transactions on Pattern Analysis and Machine Intelligence. 2005;27(2).