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On the Effective Dimension of Light Transport

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Abstract

Light transport is often characterized within a high-dimensional space although practitioners have long known that it commonly behaves as a much lower-dimensional phenomenon. We study the effective dimension *of light transport over a neighborhood on the scene manifold and show that under plausible assumptions the dimensionality is characterized by the spectrum of the spatio-spectral concentration problem. This allows us to improve existing estimates for the dimension in computer graphics using a more insightful derivation and for the first time we obtain optimal representations. The relevance of our results for existing rendering applications is discussed.*

Categories and Subject Descriptors (according to ACM CCS): Computer Graphics [I.3.7]: Three-Dimensional Graphics and Realism—[Color, shading, shadowing, and texture]

1. Introduction

In many applications the light transport problem must be solved not at a single location but over a neighborhood on the scene manifold. We study the local structure of light transport and characterize the coherence of the computations dependent on the neighborhood and the signal complexity.

Consider a radially symmetric transport operator T with axis $\mathbf{c}(x)$, mapping distant incident radiance $E(\omega)$ to exitant radiance $B(x)$,

$$B(x) = \int_{\mathcal{H}^2} E(\omega)\, T(\mathbf{c}(x)\cdot\omega)\, d\omega. \tag{1}$$

For T being bandlimited in the Spherical Harmonics domain, we determine optimal linear K-term approximations $\tilde{\mathrm{T}}$ such that the approximation error

$$\|\mathrm{B} - \tilde{\mathrm{B}}\|^2_{\mathcal{U}'} = \|\mathrm{TE} - \tilde{\mathrm{T}}\mathrm{E}\|^2_{\mathcal{U}'} \tag{2}$$

over a convex neighborhood $\mathcal{U}'$ on the scene manifold $\mathcal{M}$ is minimized, cf. Fig. 1. For sufficiently large K, with the original size M of the problem much larger than K, the error in Eq. 2 becomes vanishingly small. K will then be referred to as the *effective dimension* and K/M provides a measure for the coherence of light transport in the neighborhood $\mathcal{U}'$.

The above assumptions on T are satisfied for example by the diffuse shading kernel $(\mathbf{n}(x)\cdot\omega)$ where the axis of symmetry is the local normal $\mathbf{n}(x)$ and $L = 2$ [RH02]. Specular transport can be studied with our framework when only a single outgoing direction $\tilde{\omega}$ is of interest, in which case $B(x) = B(x, \tilde{\omega})$, and the Spherical Harmonics representation is restricted to a bandlimit L, chosen in accordance with the application of interest. We will analyze partial occlusion by assuming that the lighting is also radially symmetric.

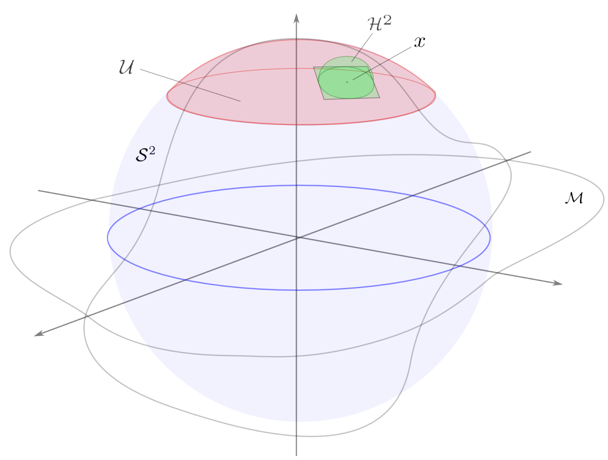

Figure 1: *We study the effective dimension of the exitant radiance $B(x)$ over the region $\mathcal{U}' \subset \mathcal{M}$ when the scene manifold $\mathcal{M}$ is locally approximated by a subset of the sphere $\mathcal{U} \subset \mathcal{S}^2_{\mathcal{M}}$.*

The local coherence of light transport has been exploited in precomputed radiance transfer for some time [SHHS03, LSSS04]. A theoretical analysis was recently undertaken by Mahajan et al. [MSRB07]. There, the problem was considered in flatland and results were obtained by estimating the spectrum of a discretized transport operator using a varia-

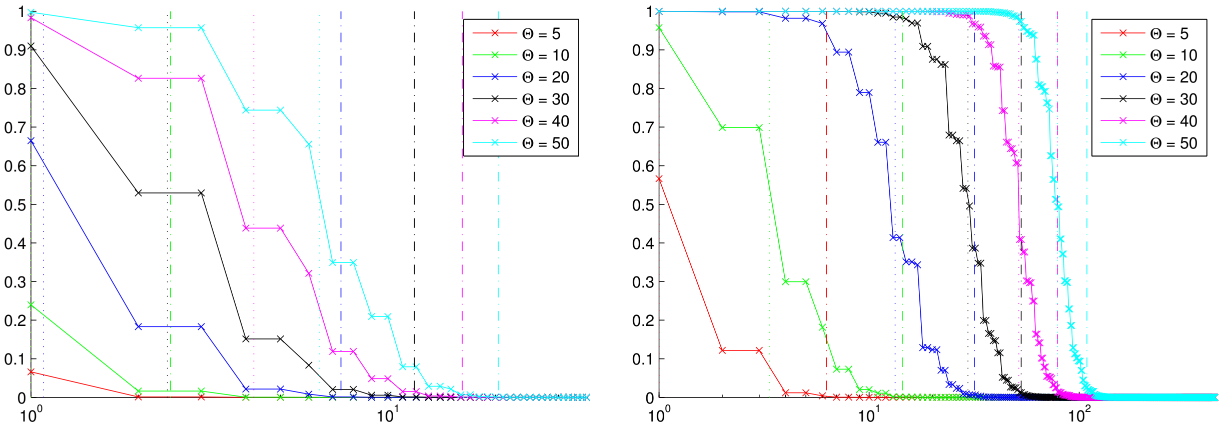

Figure 2: *Spectrum of the spatio-spectral concentration problem on the sphere for spherical caps with $\theta \leq \Theta$ and $L = 5$ (left) and $L = 20$ (right). The eigenvalue index is shown on the X axis and the magnitude $|\lambda_i|$ of the eigenvalues on the Y axis. Shown are also the Shannon number N (dotted) and generalized Shannon number $N_g(\varepsilon)$ (dash-dotted), the latter one computed with $\mathrm{B}(\partial U) = \lg\left((L+1)^2 |\partial\mathcal{U}|^2\right) / \lg(2\pi)$. Note the importance of the second term in $N_g(\varepsilon)$ when the region of spatio-spectral concentration is small.*

tion of Szegö's eigenvalue theorem. The extension to three dimensions was sketched by these authors. Using the same assumptions as Mahajan et al. [MSRB07], we study the effective dimension of light transport by reducing Eq. 2 to the spatio-spectral concentration problem on the sphere, obtaining a derivation which provides greater insight and improves their results [MSRB07].

In Sec. 2 an introduction to spatio-spectral concentration theory will be presented and in Sec. 3 we study the effective dimension of light transport. We conclude with a discussion of our results and their relevance for practical settings.

2. Spatio-Spectral Concentration on the Sphere

Let $\mathcal{H}_{\leq L}(\mathcal{S}^2)$ be the $(L+1)^2$-dimensional space of L-bandlimited functions on the two-sphere $\mathcal{S}^2$. Legendre Spherical Harmonics $y_{lm}(\omega)$ afford in many applications an effective representation for $\mathcal{H}_{\leq L}$. For signals localized in a region $\mathcal{U} \subset \mathcal{S}^2$, however, these are inefficient. Slepian functions provide in this case optimal L-bandlimited representations with $K \ll (L+1)^2$ basis functions. The complete set of $(L+1)^2$ Slepian functions is additionally orthogonal over $\mathcal{U}$ and orthonormal over $\mathcal{S}^2$, thus providing an alternative basis for $\mathcal{H}_{\leq L}$.

Slepian functions g_i are the solution to the spatio-spectral concentration problem on the sphere [SDW06, SHB09]. Let

$$\lambda_i = \frac{\|g_i\|^2_{\mathcal{U}}}{\|g_i\|^2_{\mathcal{S}^2}} = \frac{\int_U g_i^2 \, d\omega}{\int_{\mathcal{S}^2} g_i^2 \, d\omega} \tag{3}$$

be a measure for the concentration of the L-bandlimited function $g_i \in \mathcal{H}_{\leq L}$ in an arbitrary but fixed region $\mathcal{U} \subset \mathcal{S}^2$. By the Fourier uncertainty principle on the sphere [FGS98, Theorem 5.5.1] no bandlimited function in $\mathcal{H}_{\leq L}$ can be localized on a subset of the sphere. Hence $\lambda_i < 1$. Functions $g_i \in \mathcal{H}_{\leq L}$ which are optimally localized according to Eq. 3 are given by the eigenfunctions of the integral equation

$$\int_{\mathcal{U}} D(\bar{\omega} \cdot \omega) \, g_i(\omega) \, d\omega = \lambda_i \, g_i(\bar{\omega}) \tag{4}$$

whose kernel $D(\bar{\omega} \cdot \omega) = \sum_{l=0}^{L} ((2l+1)/4\pi) \; P_l(\bar{\omega} \cdot \omega)$ is formed by the Zonal Harmonics $P_l(\bar{\omega} \cdot \omega)$; numerical solutions for the g_i can be obtained in the Spherical Harmonics domain where Eq. 4 reduces to an $(L+1) \times (L+1)$ dimensional matrix eigenvalue problem. The integral operator $D(\bar{\omega} \cdot \omega)$ is symmetric and positive. Hence there are $n = (L+1)^2$ eigenvalues $1 > \lambda_1 \geq \ldots \geq \lambda_n > 0$ and the associated eigenfunctions g_i can be chosen to be orthogonal. The first eigenfunction g_1 is the L-bandlimited function which is optimally concentrated in $\mathcal{U}$ and by construction its localization is λ_1. Subsequent eigenfunctions g_i are the maximally concentrated functions in $\mathcal{H}_{\leq L}$ orthogonal to all g_j with $j < i$.

Of practical interest is the number K of Slepian functions which is needed to accurately represent a signal localized in $\mathcal{U} \subset \mathcal{S}^2$. It follows from the definition of the concentration measure in Eq. 3 that basis functions g_i with very small eigenvalues $|\lambda_i| \approx 0$ practically vanish over $\mathcal{U}$ and their contribution to a representation of a $\mathcal{U}$-localized signal is negligible. An accurate approximation can hence be obtained by considering only Slepian functions g_i whose eigenvalues λ_i differ significantly from zero. The magnitude of the eigenvalues is described by the localization spectrum of the spatio-spectral concentration problem. For the real line it has been shown [Sle65, Lan65, LW80] that the number $N(\varepsilon)$ of eigenvalues greater than $\varepsilon > 0$ satisfies

$$N(\varepsilon) = \frac{\tilde{C}}{\pi} + \log\left(\frac{1-\varepsilon}{\varepsilon}\right) \frac{\log(\tilde{C})}{\pi^2} + o(\log(\tilde{C})), \tag{5}$$

where $\tilde{C} = \tilde{L}\tilde{U}$ is the spatio-spectral region of concentration for bandlimited $\tilde{L}$ and an interval of size $\tilde{U}$. The three terms in Eq. 5 are characteristic for the localization spectrum: The first term counts the number of eigenvalues close to unity, the second term determines the size of a region where the eigenvalue magnitude decays exponentially from one to zero, and the last term represents eigenvalues close to zero. For a sufficiently large region $\tilde{C}$ the first term is dominant and it pro-

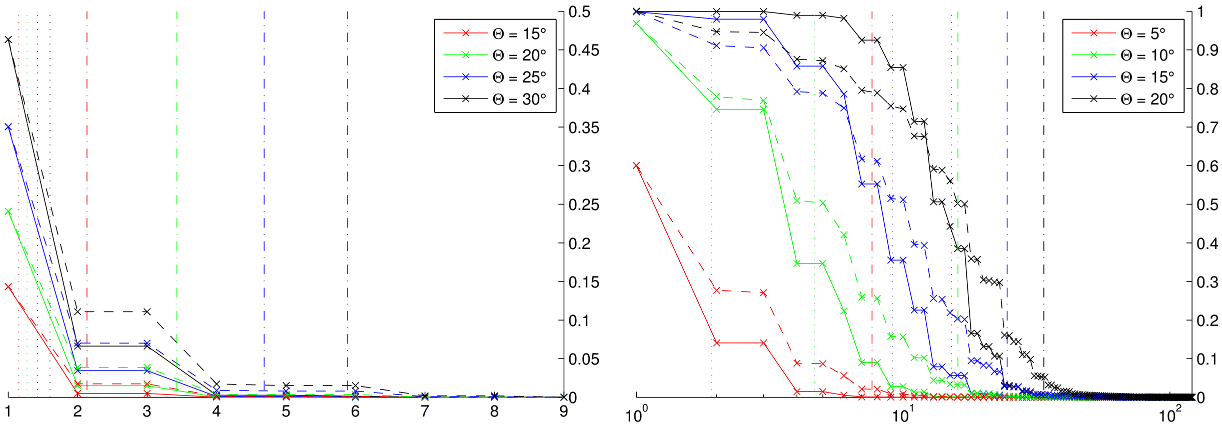

Figure 3: *Spectrum of the spatio-spectral concentration problem (full) and for empirical transport operators (dashed, obtained using singular value decomposition) for spherical caps with $\theta \leq \Theta$. Left, diffuse transport $T(\mathbf{n}(x), \omega) = (\mathbf{n}(x) \cdot \omega)$ for $L = 2$; right, Phong transport $T(\mathbf{r}(x), \omega) = (\mathbf{r}(x) \cdot \omega)^{128}$ for $L = 10$, $\mathbf{r}(x)$ is the local reflection direction. Formatting as in Fig. 2. Differences in the spectra arise from the non-uniform energy distribution for the empirical transport operators across bands.*

vides an accurate estimate for the number of non-vanishing eigenvalues. For small $\tilde{C}$ both the first and the second term have to be considered. Unfortunately, for the sphere no results analogous to Eq. 5 have been established to date. However, the analogy to Eq. 5 and a setting studied by Slepian [Sle64] as well as experimental evidence [SDW06] suggest [Sim10]

$$N(\varepsilon) = \frac{C}{4\pi} + \log\left(\frac{1-\varepsilon}{\varepsilon}\right) \mathrm{B}(\partial\mathcal{U}) \log(C) + o(\log C), \qquad (6)$$

where $\mathrm{B}(\partial\mathcal{U})$ is a function which depends on the boundary $\partial\mathcal{U}$ of the region $\mathcal{U}$, and the spatio-spectral region of concentration is $C = (L+1)^2 A(\mathcal{U})$ with area $A(\mathcal{U})$. The three terms in Eq. 6 correspond again to the three characteristic parts of the localization spectrum. The first term is known as the *spherical Shannon number* N and we will refer to the first two terms as the *generalized spherical Shannon number* N_g. A restricted setting which will be of interest in the following is when $\mathcal{U}$ is a spherical cap of co-latitude Θ and only radially symmetric Slepian functions are considered. The spectrum is then again described by Eq. 6 but the spatio-spectral region of concentration is in this case $C = 4(L+1)\Theta$, cf. [WS05].

Sample spectra of the spatio-spectral concentration problem are shown in Fig. 2. Additional spectra and visualizations of the basis functions are available in [LF10, SDW06].

3. The Effective Dimension of Light Transport

We will now return to our study of the effective dimension of light transport. Consider Eq. 1 and let the transport operator be L-bandlimited. Furthermore, assume that $\mathbf{c}(x)$ is the local normal $\mathbf{n}(x)$; we will discuss generalizations in the sequel. By its radial symmetry, $T(\mathbf{n}(x) \cdot \omega)$ is then naturally represented using Zonal Harmonics $P_l(\mathbf{n}(x) \cdot \omega)$ centered at $\mathbf{n}(x)$,

$$T(\mathbf{n}(x) \cdot \omega) = \sum_{l=0}^{L} t_l \, P_l(\mathbf{n}(x) \cdot \omega) = \sum_{l=0}^{L} t_l \sum_{m=-l}^{l} y_{lm}(\mathbf{n}(x)) \, y_{lm}(\omega),$$

where the right hand side follows from the Spherical Harmonics addition theorem. By expanding the incident radiance $E(\omega)$ in Spherical Harmonics and with the domain of integration extended to $\mathcal{S}^2$, the domain of orthonormality of the y_{lm}, we can rewrite Eq. 1 in a global coordinate frame as

$$\begin{aligned} B(x) &= \int_{\mathcal{S}^2} \left(\sum_{l'=0}^{L} \sum_{m'=-l'}^{l'} e_{l'm'} \, y_{l'm'}(\omega) \sum_{l=0}^{L} t_l \, P_l(\mathbf{n}(x) \cdot \omega) \right) d\omega \\ &= \sum_{l'=0}^{L} \sum_{m'=-l'}^{l'} e_{l'm'} \sum_{l=0}^{L} \sum_{m=-l}^{l} t_l \, y_{lm}(\mathbf{n}(x)) \int_{\mathcal{S}^2} y_{l'm'} \, y_{lm} \, d\omega \\ &= \sum_{l=0}^{L} \sum_{m=-l}^{l} e_{lm} \, t_l \, y_{lm}(\mathbf{n}(x)). \end{aligned} \qquad (7)$$

For a convex region $\mathcal{U}'$ on $\mathcal{M}$ which is well approximated by a subset $\mathcal{U} \subset \mathcal{S}^2_{\mathcal{M}}$ of the sphere $\mathcal{S}^2_{\mathcal{M}}$, the local normal $\mathbf{n}(x)$ coincides with the direction vector $\bar{\omega}$ of $\mathcal{S}^2_{\mathcal{M}}$, cf. Fig. 1. Eq. 7 can then be written as

$$B(x) \approx B(\bar{\omega}) = \sum_{l=0}^{L} \sum_{m=-l}^{l} e_{lm} \, t_l \, y_{lm} = \sum_{l=0}^{L} \sum_{m=-l}^{l} b_{lm} \, y_{lm}(\bar{\omega}) \qquad (8)$$

where $B(x)$ defined over $\mathcal{M}$ is approximated by the L-bandlimited signal $B(\bar{\omega})$ represented in Spherical Harmonics $y_{lm}(\bar{\omega})$ defined over $\mathcal{S}^2_{\mathcal{M}}$. In the following we will be concerned with an optimal K-term approximation of $B(\bar{\omega})$ and demonstrate that it is given by Slepian functions.

Let $\{\varphi_i\}_{i=1}^{M}$ with $M = (L+1)^2$ be an arbitrary basis for $\mathcal{H}_{\leq L}(\mathcal{S}^2_{\mathcal{M}})$. A linear K-term approximation of Eq. 8 is obtained when $K \leq M$ basis functions $\varphi_i(\bar{\omega})$ with $i \in \Lambda$ are employed to represent $B(\bar{\omega})$ and the index set Λ is chosen irrespective of the signal. Assuming the φ_i are orthogonal over $\mathcal{U}$, the approximation error for the exitant radiance $B(\bar{\omega})$ over $\mathcal{U}$ in Eq. 2 can be written as

$$\|B(\bar{\omega}) - \tilde{B}(\bar{\omega})\|_{\mathcal{U}} = \sum_{i=K+1}^{M} \bar{b}_i^2 \, \langle \varphi_i, \varphi_i \rangle_{\mathcal{U}} = \sum_{i=K+1}^{M} \bar{b}_i^2 \, \|\varphi_i\|_{\mathcal{U}}^2.$$

For arbitrary E and T, the error is thus minimized by basis functions $\varphi_i(\bar{\omega})$ whose squared norm $\|\varphi_i\|_{\mathcal{U}}^2$ is minimal over $\mathcal{U}$. However, the K bandlimited functions in $\mathcal{H}_{\leq L}$ whose norm is minimal over $\mathcal{U}$ are the *last* $M-K$ Slepian functions. An optimal K-term approximation for $B(\bar{\omega})$ is hence provided by the first K Slepian functions and these naturally satisfy our previous orthogonality assumption. The sought estimate for the effective dimension of T in a neighborhood $\mathcal{U}'$ is then provided by the Shannon number N and the generalized Shannon number N_g.

To study the effective dimension of light transport in partially occluded environments we will assume that $E(\omega) = E(\mathbf{h}\cdot\omega)$ is radially symmetric with axis $\mathbf{h} \in \mathcal{S}^2$ and consider the setup in Fig. 4. In the hemisphere above the origin the lighting is then symmetric around the up axis and we assume that the neighborhood $\mathcal{U}$ is sufficiently small so that deviations from this assumption are negligible for all $x \in \mathcal{U}$. With a derivation analogous to those which led to Eq. 7 one obtains

$$B(x) = \sum_{l=0}^{L} e_l\, t_l\, P_l(\mathbf{h}\cdot\mathbf{n}(x)). \approx \sum_{l=0}^{L} e_l\, t_l\, P_l(\mathbf{h}\cdot\bar{\omega}). \qquad (9)$$

where we again identified $\mathbf{n}(x)$ and $\bar{\omega} \in \mathcal{S}^2_{\mathcal{M}}$. i It follows that the best K-term approximation for $B(\bar{\omega})$ over all of $\mathcal{U}$ is given by the L-bandlimited and radially symmetric functions which are optimally concentrated in $\mathcal{U}$. Slepian function hence once again provide the optimal representation. If we furthermore assume that $\mathcal{U}$ is a spherical cap of co-latitude Θ, then by the radial symmetry of the $P_l(\mathbf{h}\cdot\bar{\omega})$ in Eq. 9 the spatio-spectral region of concentration is $C = 4\,(L+1)\,\Theta$. For radial symmetric lighting and a region of concentration which is a spherical cap the dimensionality hence depends directly on the co-latitude Θ instead of the area $A(\mathcal{U})$.

Experimental results are presented in Fig. 3. Note the importance of the second term of the generalized Shannon number for accurate estimates of the dimensionality. Differences between the theoretical and empirical spectra arise from magnitude variations of the basis function coefficients for real transport operators which we do not model.

4. Discussion

We showed that the effective dimension of a bandlimited and radially symmetric transport operator in a local neighborhood on the scene manifold can be studied using spatio-spectral concentration theory. An estimate for the dimensionality is given by the Shannon number N and a more refined analysis

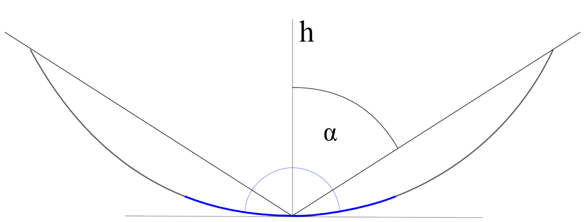

Figure 4: *To study partial occlusion we consider the interior of a hemispherical cap (shown is a cut) and a neighborhood $\mathcal{U}$ (blue) centered at the pole, similar to the setup by Ramamoorthi et al. [RKB05].*

leads to the generalized Shannon number N_g. This demonstrates that there exists both a linear *and* a logarithmic dependence on the region of concentration. The logarithmic term was not obtained in previous work [MSRB07] although it is of particular importance for the settings considered in computer graphics where very low bandlimits L are employed. For radially symmetric lighting and a neighborhood $\mathcal{U}$ which is a spherical cap, a setting which allows one to study partial occlusion, we showed that the spatio-spectral region of concentration depends linearly on the co-latitude Θ of the spherical cap instead of the area $A(\mathcal{U})$, paralleling the result by Mahajan et al. [MSRB07]. In addition to the refined estimates for the effective dimension, the results available in spatio-spectral concentration theory are more general than those obtained previously; for example, they hold for arbitrarily shaped regions $\mathcal{U} \subset \mathcal{S}^2$ and provide insight into the eigenfunctions associated with the problem. Our results also do not rely on a discretization of the transport operator. Unfortunately, the eigenvalue distribution for the spatio-spectral concentration problem on the sphere in Eq. 6 remains a conjecture and the true boundary function $\mathrm{B}(\partial U)$ is unknown. However, our experimental results demonstrate the usefulness of our novel boundary term and we believe it will have applications in other areas where Slepian functions are employed.

We studied the effective dimension of light transport by approximating the local neighborhood $\mathcal{U}' \subset \mathcal{M}$ on the scene manifold by a subset $\mathcal{U} \subset \mathcal{S}^2$ of the sphere. Other choices for $\mathcal{U}$ are conceivable and a subset $\mathcal{U} \subset \mathcal{R}^2$ of the plane is sensible in particular for regions $\mathcal{U}' \subset \mathcal{M}$ where curvature is negligible. In the plane the spectrum of the spatio-spectral concentration problem consists again of the three characteristic parts [Sle64]. Furthermore, Simons et al. [SDW06] showed that asymptotically for $A(\mathcal{U}) \to 0$ the spatio-spectral concentration problem on the sphere reduces to that in the plane. The consistency of these results is important since any estimate for the effective dimension should be independent of the details of the approximations.

In Sec. 3 we derived our result with the axis of radial symmetry being the local normal $\mathbf{n}(x)$. Our derivation is more generally applicable as long as $\mathbf{c}(x)$ can be identified uniquely with a point on the sphere. For example for the Phong operator considered by Mahajan et al. [MSRB07], with the reflection direction as the axis of symmetry, the required identification is $(\theta,\phi) = (2\,\theta,\phi)$. Most physically motivated bidirectional reflection distribution functions are not strictly radially symmetric as we assumed in our derivation. The phenomenological success of the Phong model suggests however that radial symmetry is a useful first order assumption, and for the Torrance-Sparrow model it is known that radial symmetry holds for small outgoing angles [RH04]. Further investigations would be a valuable addition to the literature.

The work by Mahajan et al. [MSRB07] was an important contribution to the literature. However, we believe that our ansatz, which states the objective as an approximation problem, is more amenable to extensions. For example, Spherical Harmonics are well-suited for the representation of smooth signals. The signals encountered in rendering are however only piece-wise smooth and hence wavelet-like constructions provide optimal representations [Mal09]. An analysis of the present problem using such bases and nonlinear approximation strategies is an exciting area for future work.

Mahajan et al. [MSRB07] employed their results for instance to guide transport matrix compression for precomputed radiance transfer and to develop object representations suited for efficient rendering. These are also possible applications for our work and our improved estimates might afford increased efficiency. In contrast to previous work, our derivation yields expressions for the basis functions and a representation using Slepian functions can be used to exploit the coherence of light transport without a costly principal component analysis. The experimental results in Fig. 3 moreover demonstrate that substantial efficiency gains can be expected with this ansatz. The required mapping from the neighborhood $\mathcal{U}' \subset \mathcal{M}$ on the scene manifold to a region $\mathcal{U} \subset \mathcal{S}^2_{\mathcal{M}}$ on the sphere can be accomplished using normal directions or the isometric embedding provided by the exponential map. A comparison to clustered principal component analysis, which was employed in previous work, is an interesting area for future work. We believe Slepian functions will also prove useful for the effective representation of localized signals such as functions defined over the hemisphere where alternatives to Spherical Harmonics have already been explored [Mak96, GKPB04, RKB05] but these often do not provide orthogonality over $\mathcal{U}$ and $\mathcal{S}^2$ and closure under rotation. For example, in precomputed radiance transfer Slepian functions would make it possible to increase the bandlimit of the signals without degrading visual quality or performance. Rotation of Slepian functions would be possible with a variation of a rotation algorithm recently developed by us for Spherical Harmonics [LdWF10]. For sampling based techniques the coherence of light transport has not been exploited systematically in the past. We believe that the understanding gained in this note will be useful to improve the efficacy of these techniques by facilitating the development of rigorous algorithms which exploit coherence between samples.

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