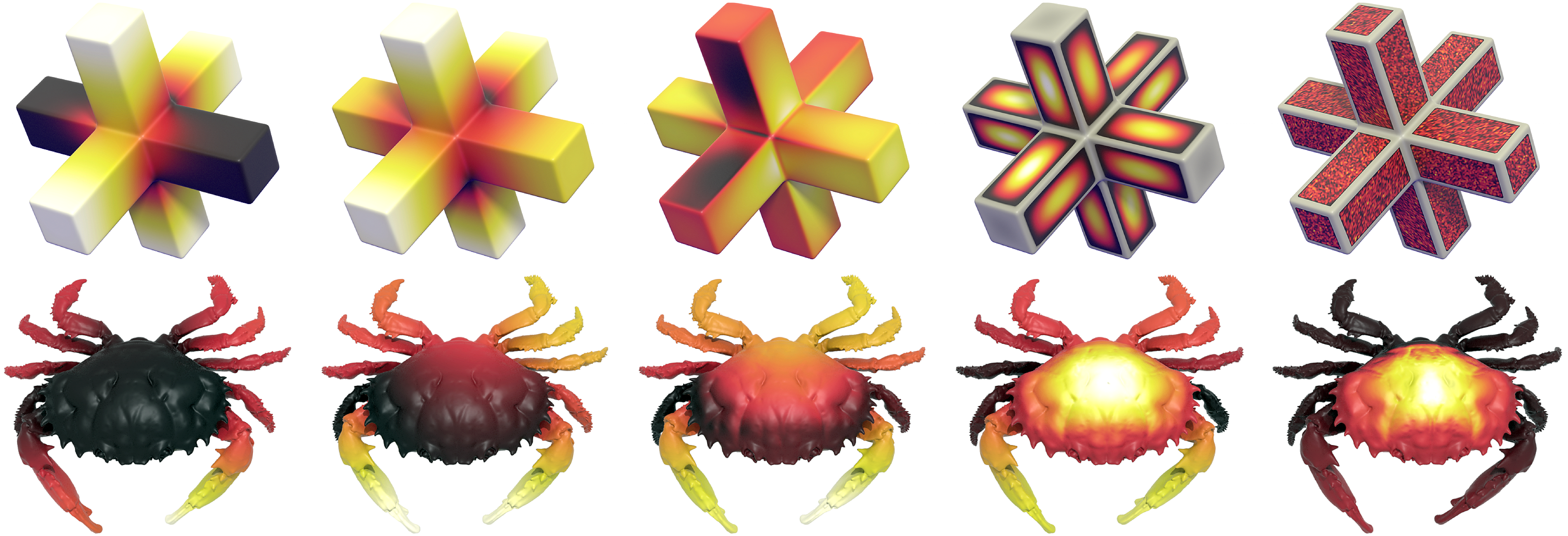


A Dirac Operator for Extrinsic Shape Analysis

Hsueh-Ti Derek Liu[1], Alec Jacobson[2], Keenan Crane[1]

[1]Carnegie Mellon University

[2]University of Toronto

Outline

Goal: extend spectral geometry processing

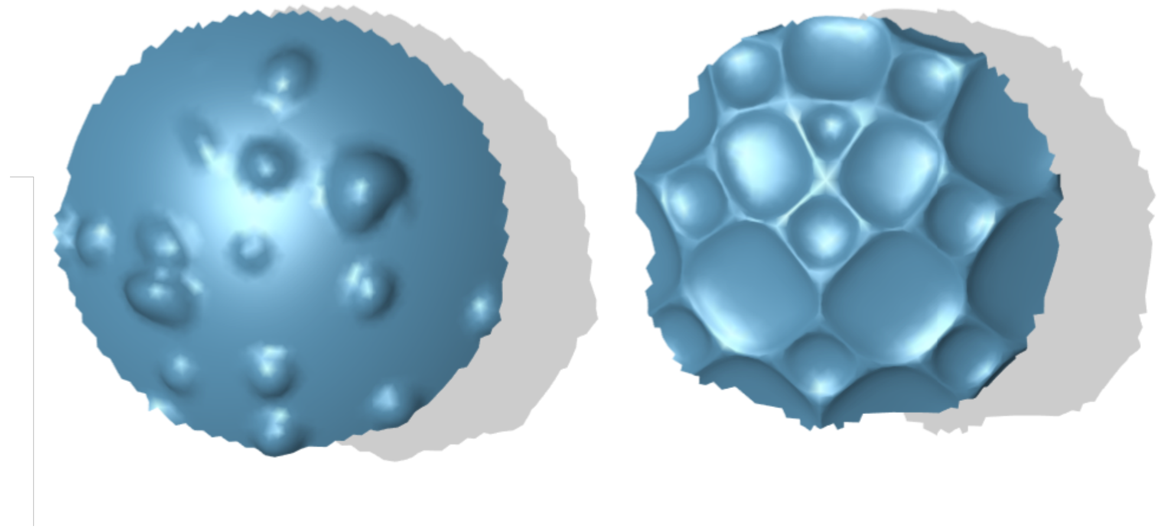
- Traditionally: *intrinsic* only (point-to-point distance)
- Today: *extrinsic* information (bending in space)

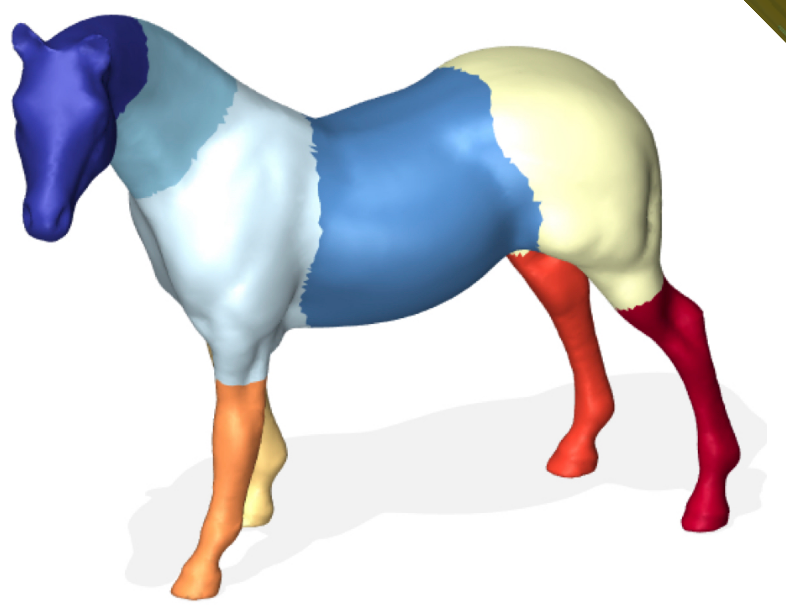
Basic idea: develop new differential operators

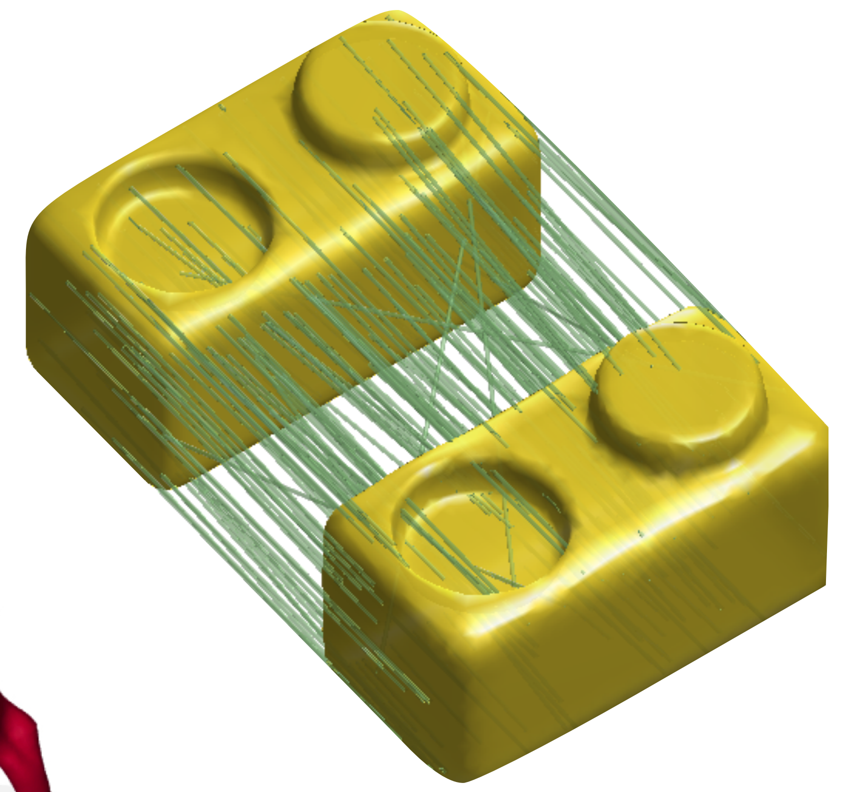
- Instead of standard Laplacian, use *relative Dirac* operator

Applications:

- Classification, segmentation, correspondence







What is Spectral Geometry Processing?

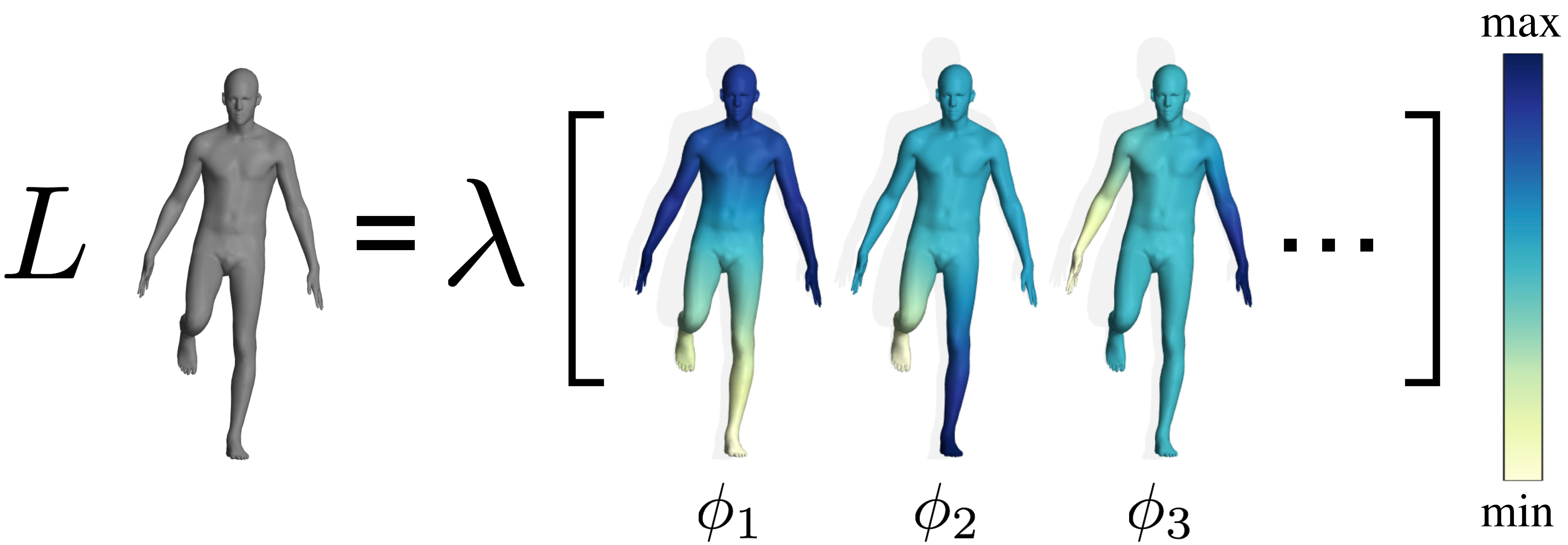
$$L\phi_i = \lambda_i \phi_i$$

differential operator

eigenvalues

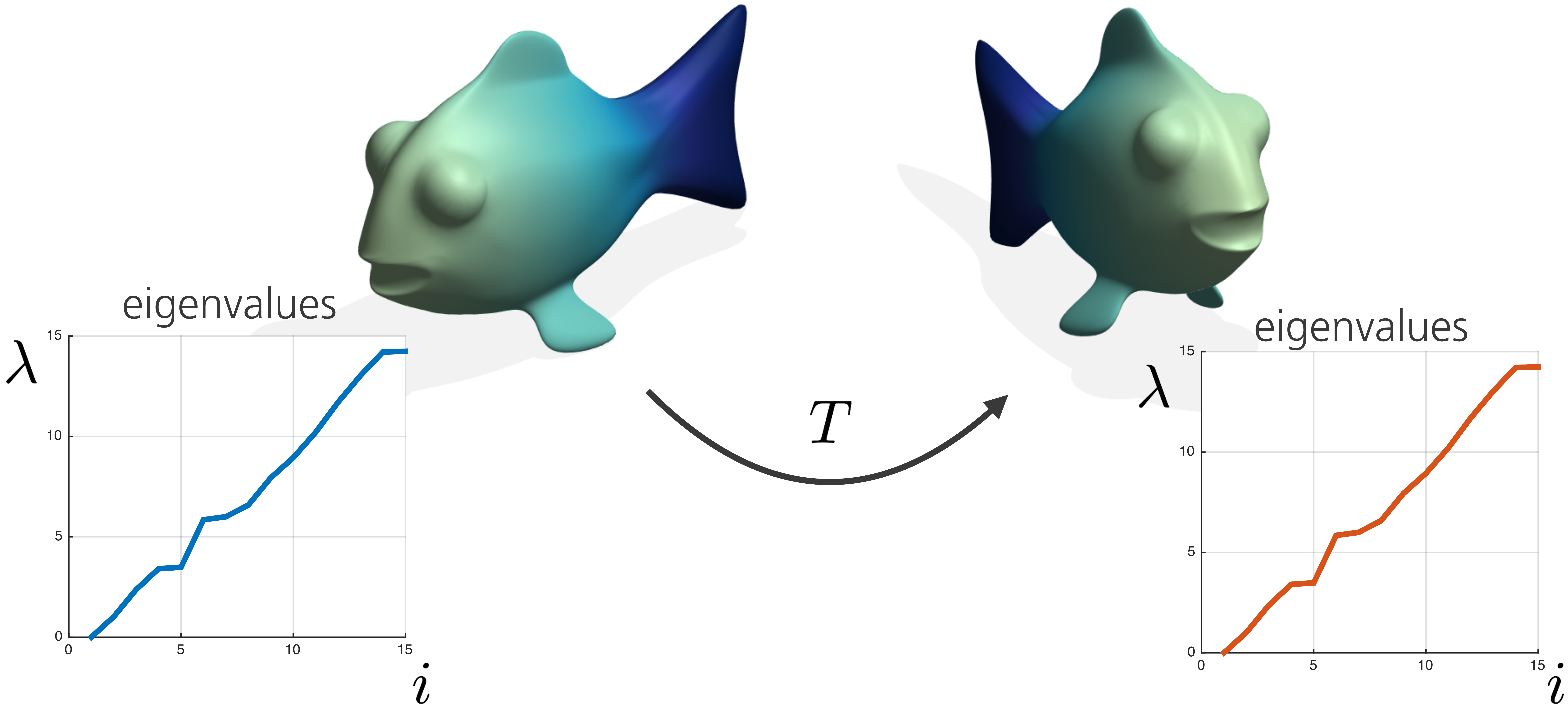
eigenvectors

What is Spectral Geometry Processing?

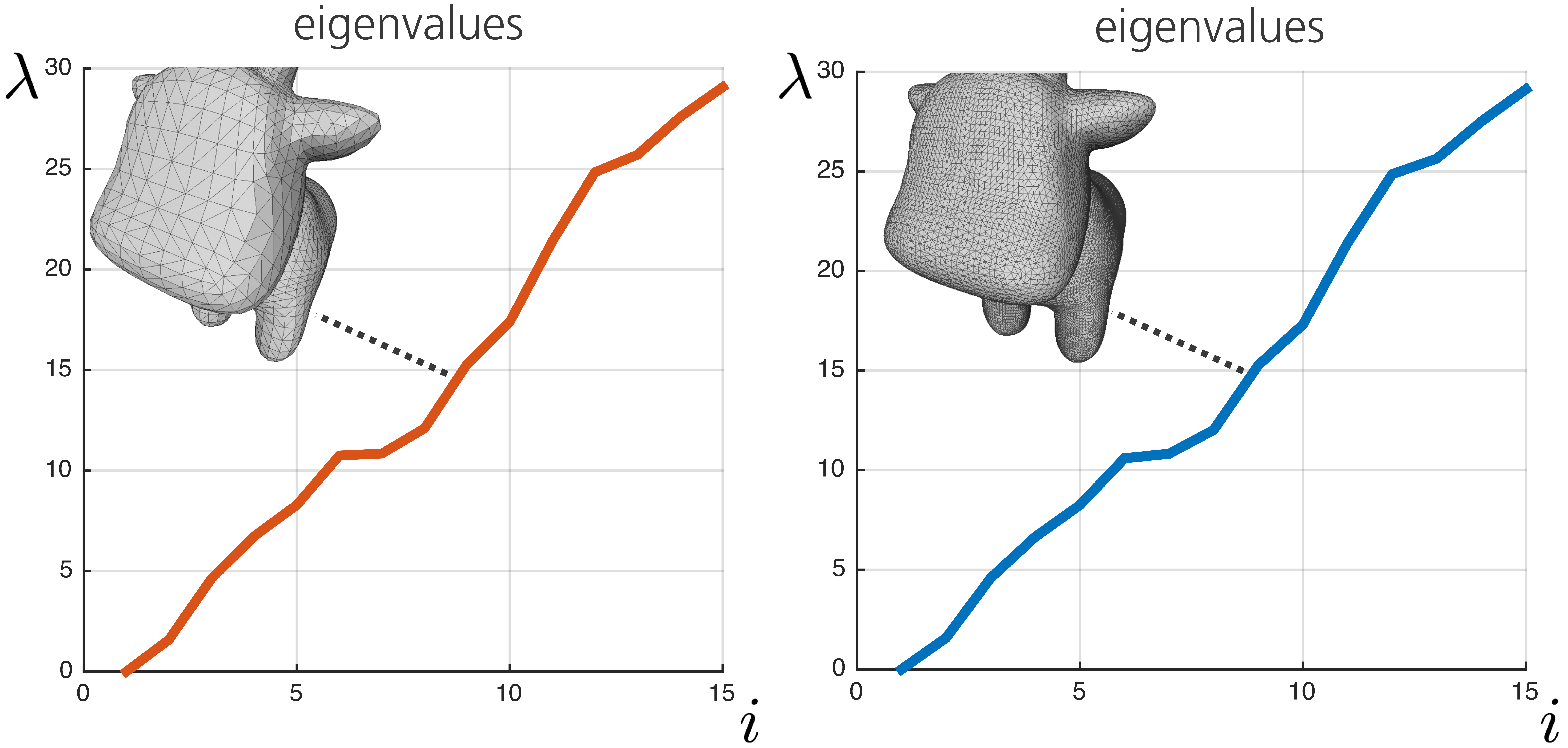


Analogy: "Fourier transform" for surfaces

Why - Coordinate Invariant



Why - (Almost) Invariant to Tessellation



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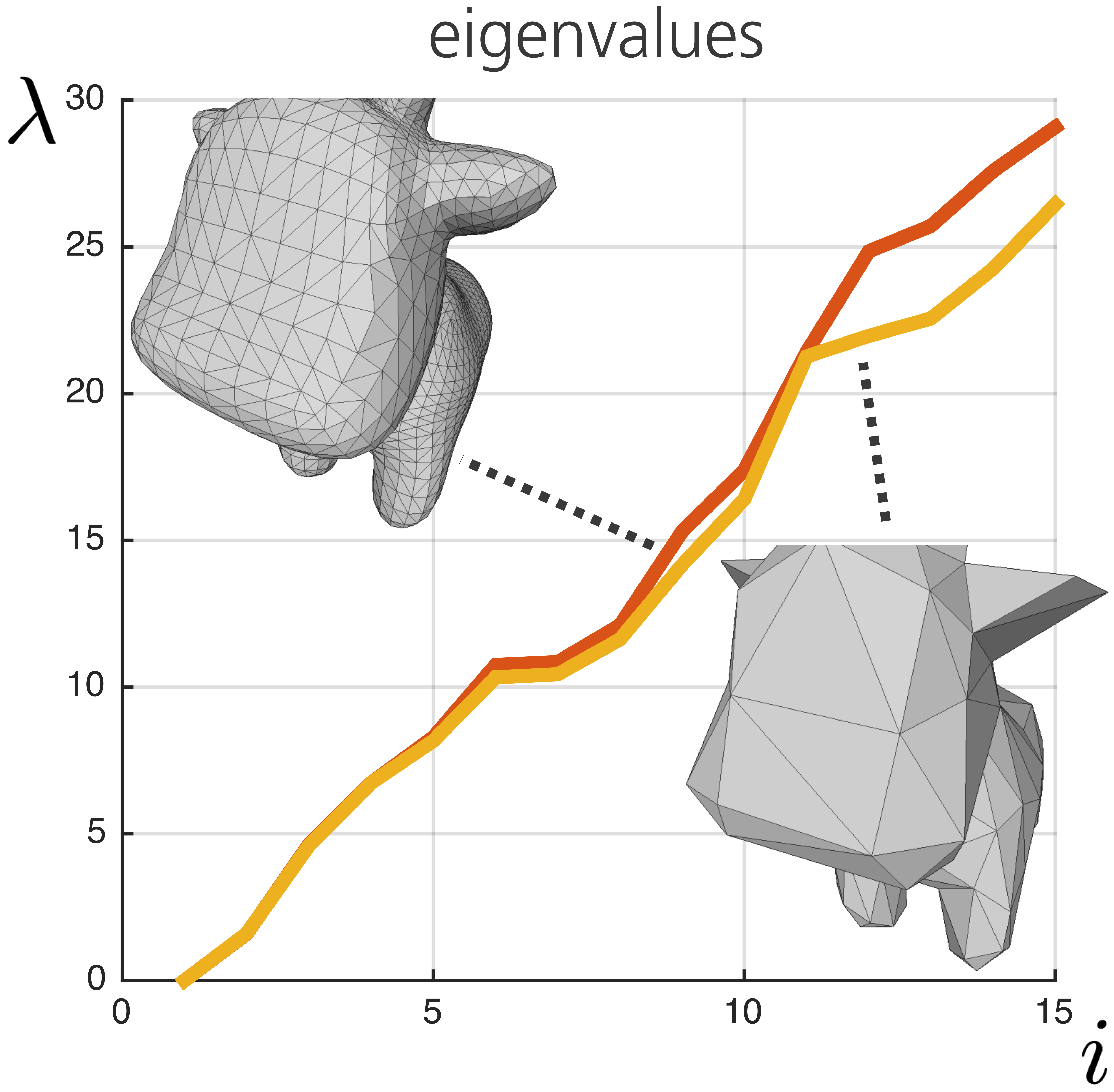
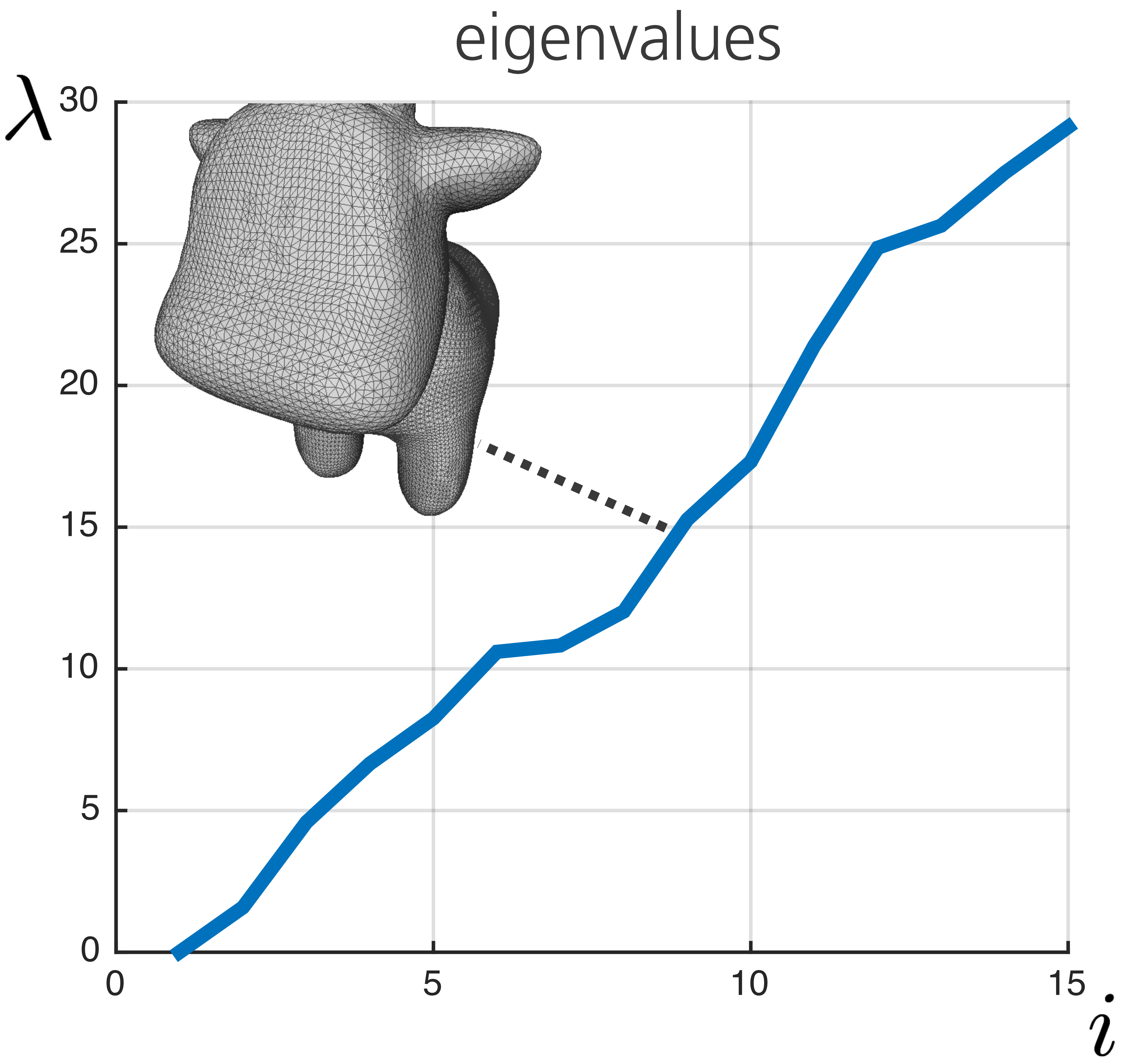
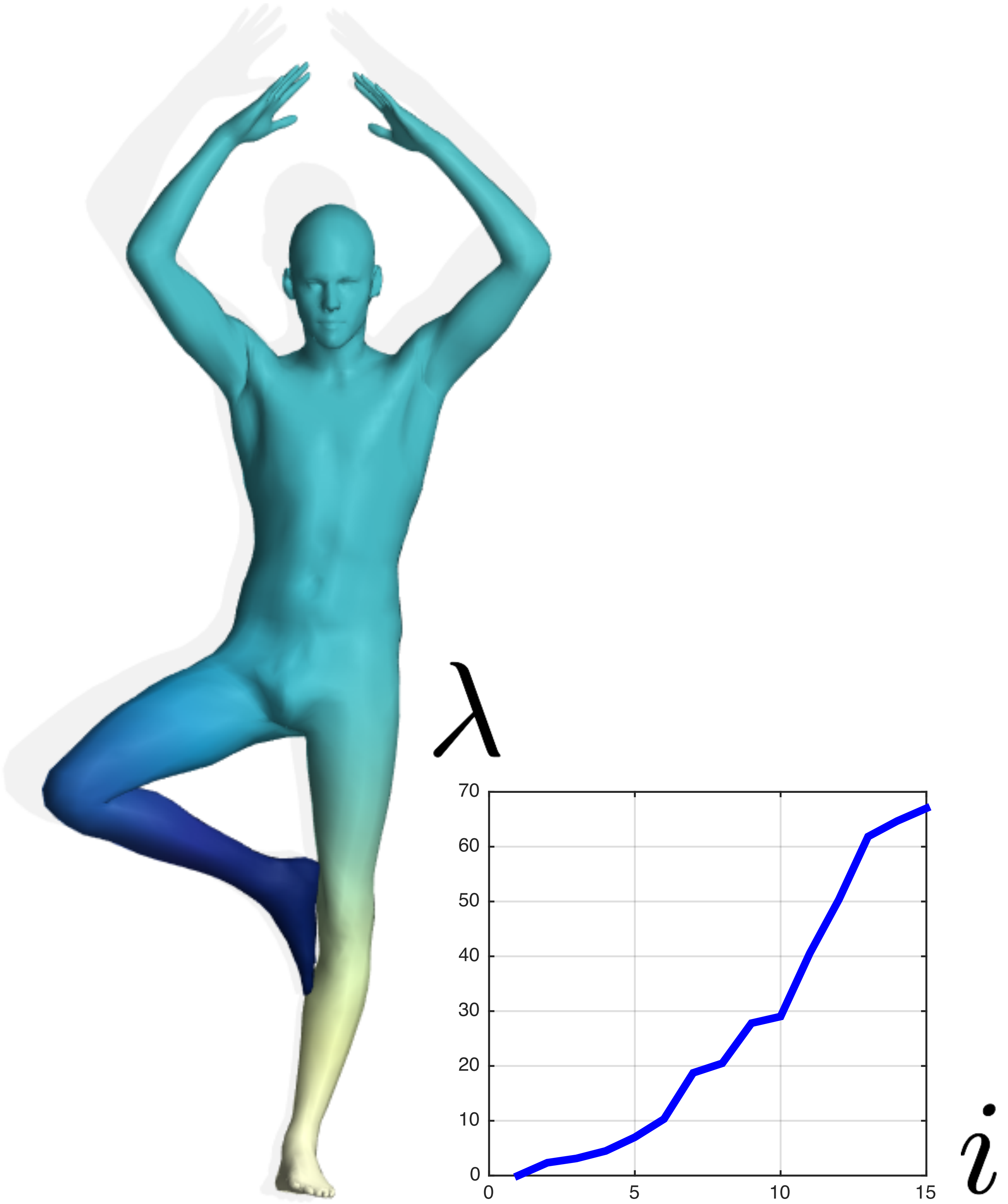
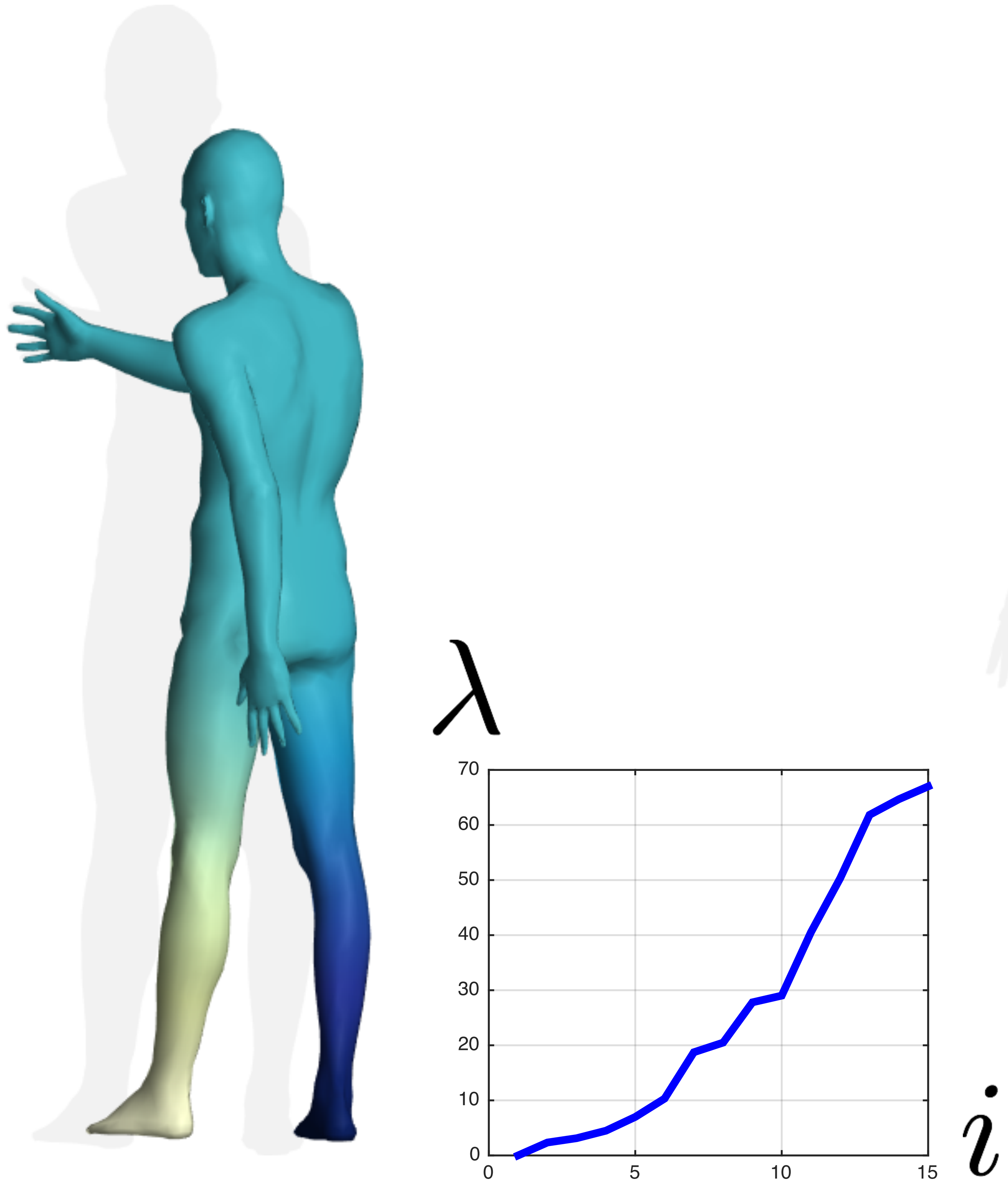


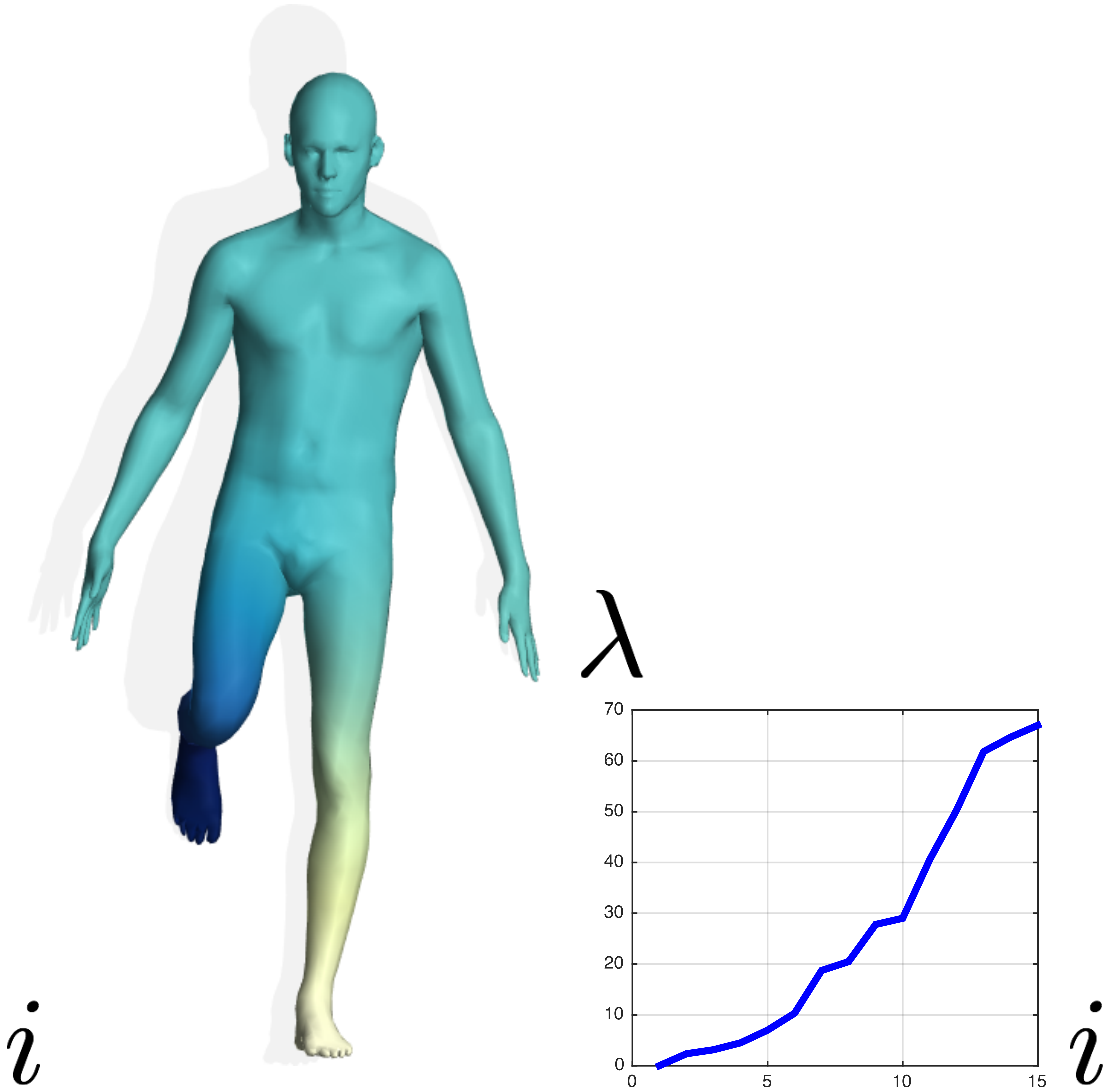


Why - Isometry Invariant

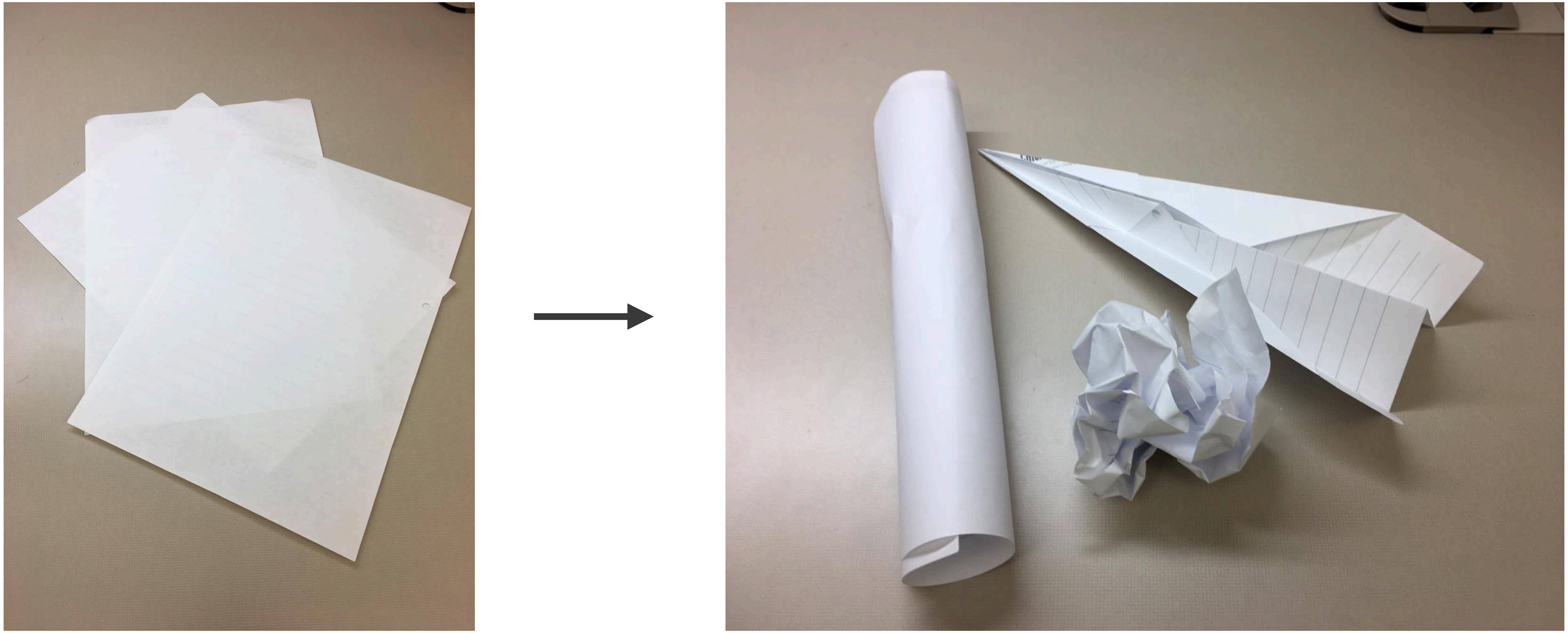
Benefits from Laplace-Beltrami operator Δ



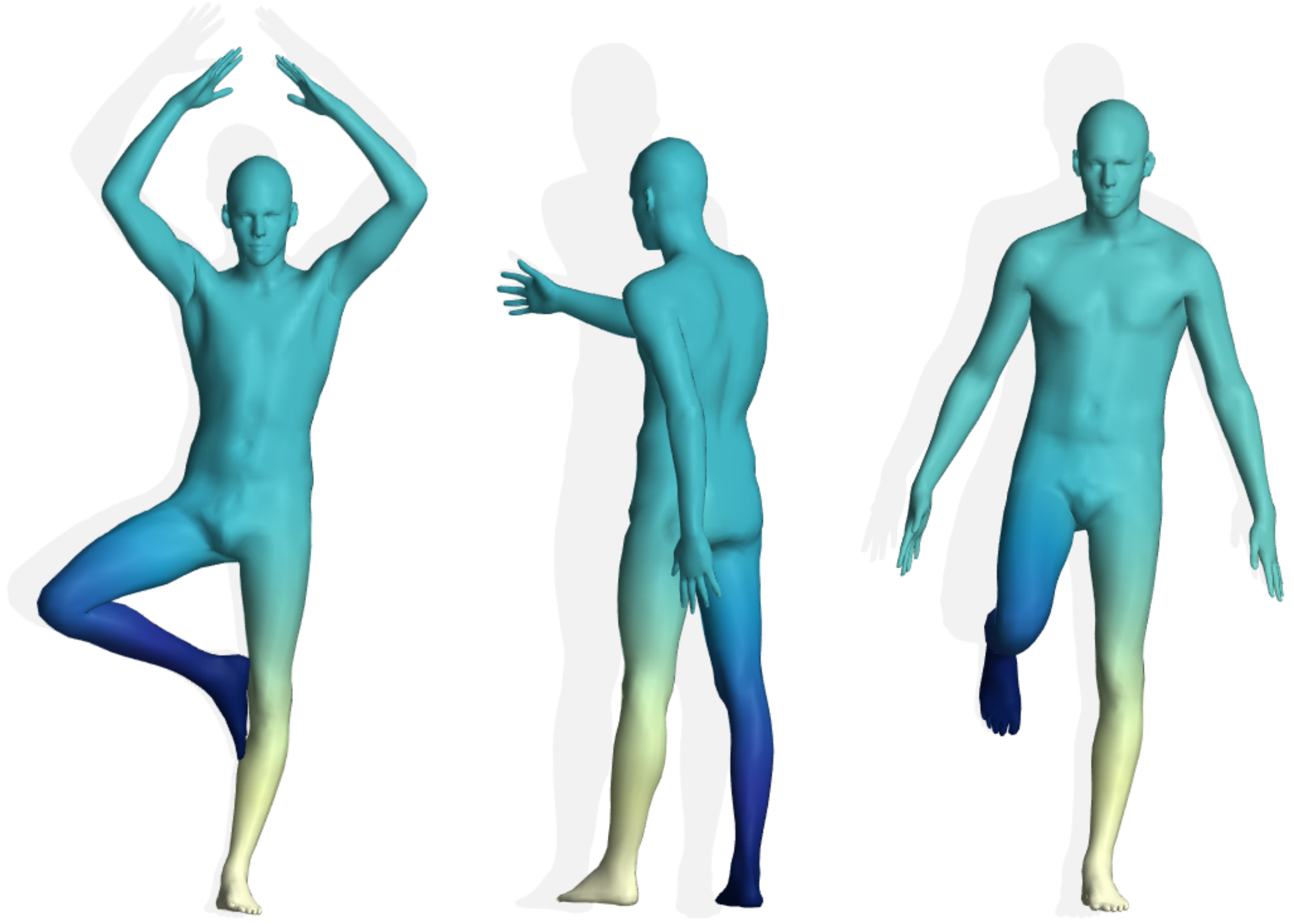


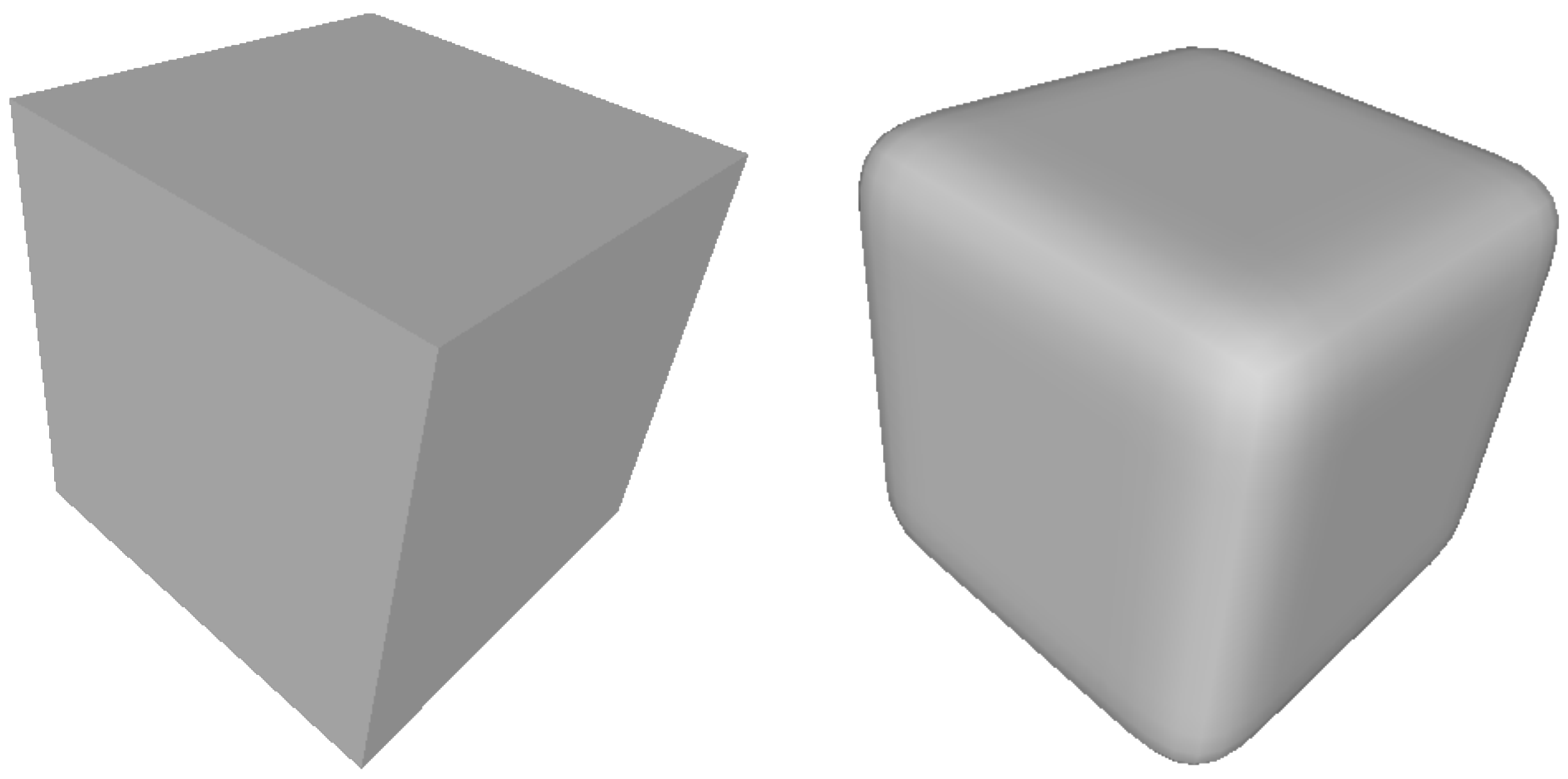


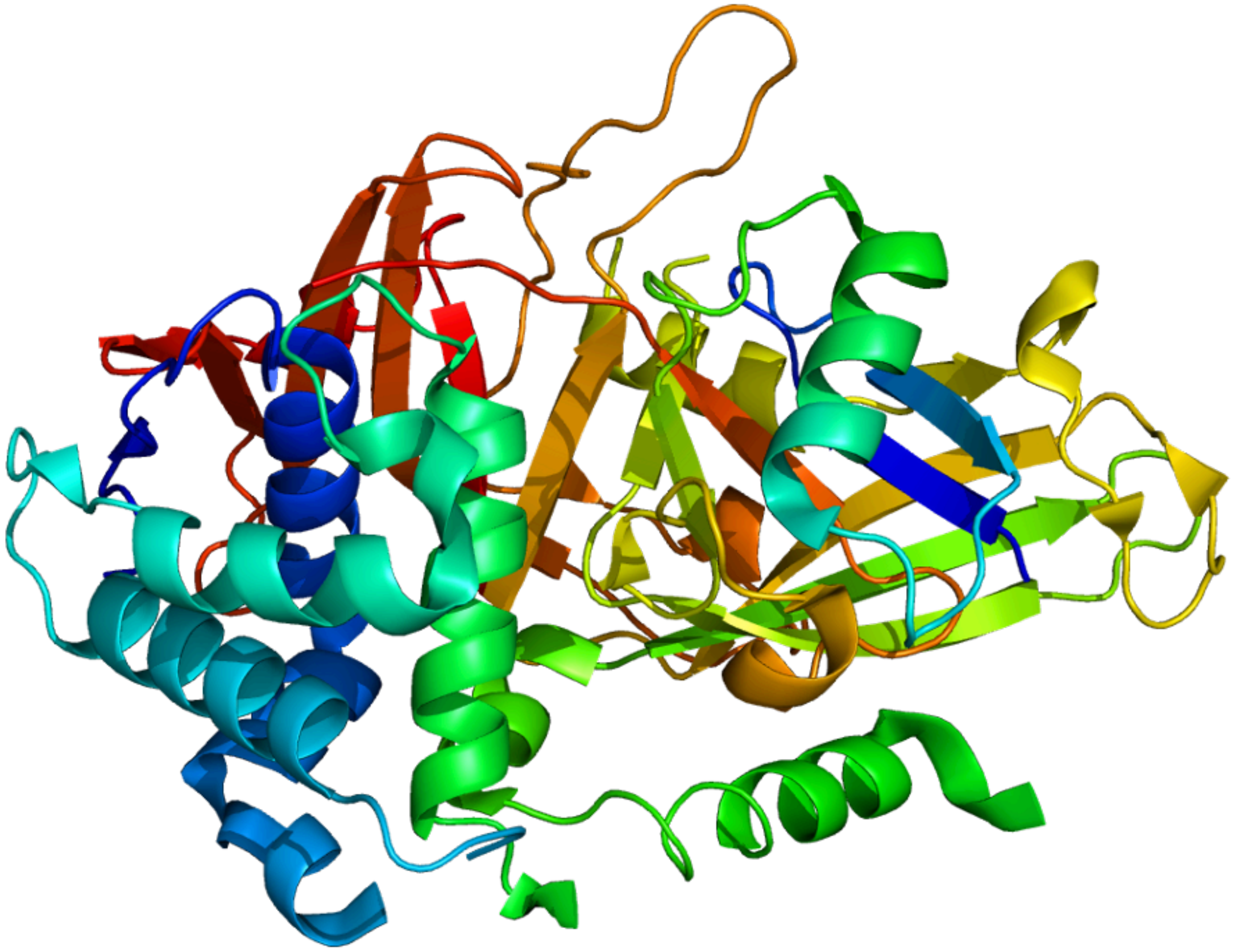
Isometry Invariance
Is it a feature or a bug?



Sensitivity to Metric Distortion







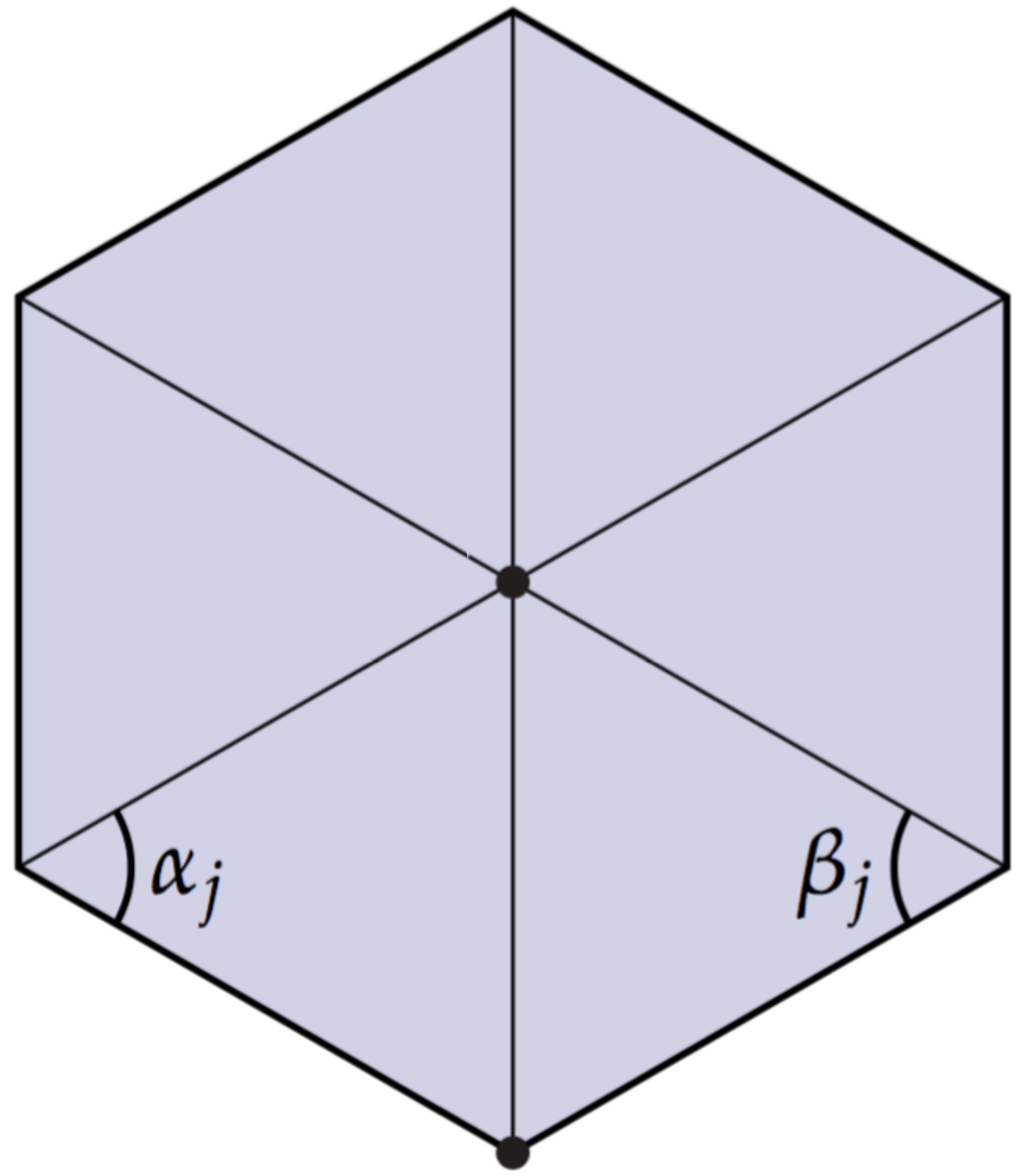
https://en.wikipedia.org/wiki/USP2

(Discrete) Differential Operators

Laplace-Beltrami Operator (intrinsic) Δ

- Discrete cotangent Laplacian

$$\Delta p_i = \frac{1}{2\mathcal{A}_i} \sum_{j \in \mathcal{N}(i)} (\cot \alpha_{ij} + \cot \beta_{ij})(p_i - p_j)$$



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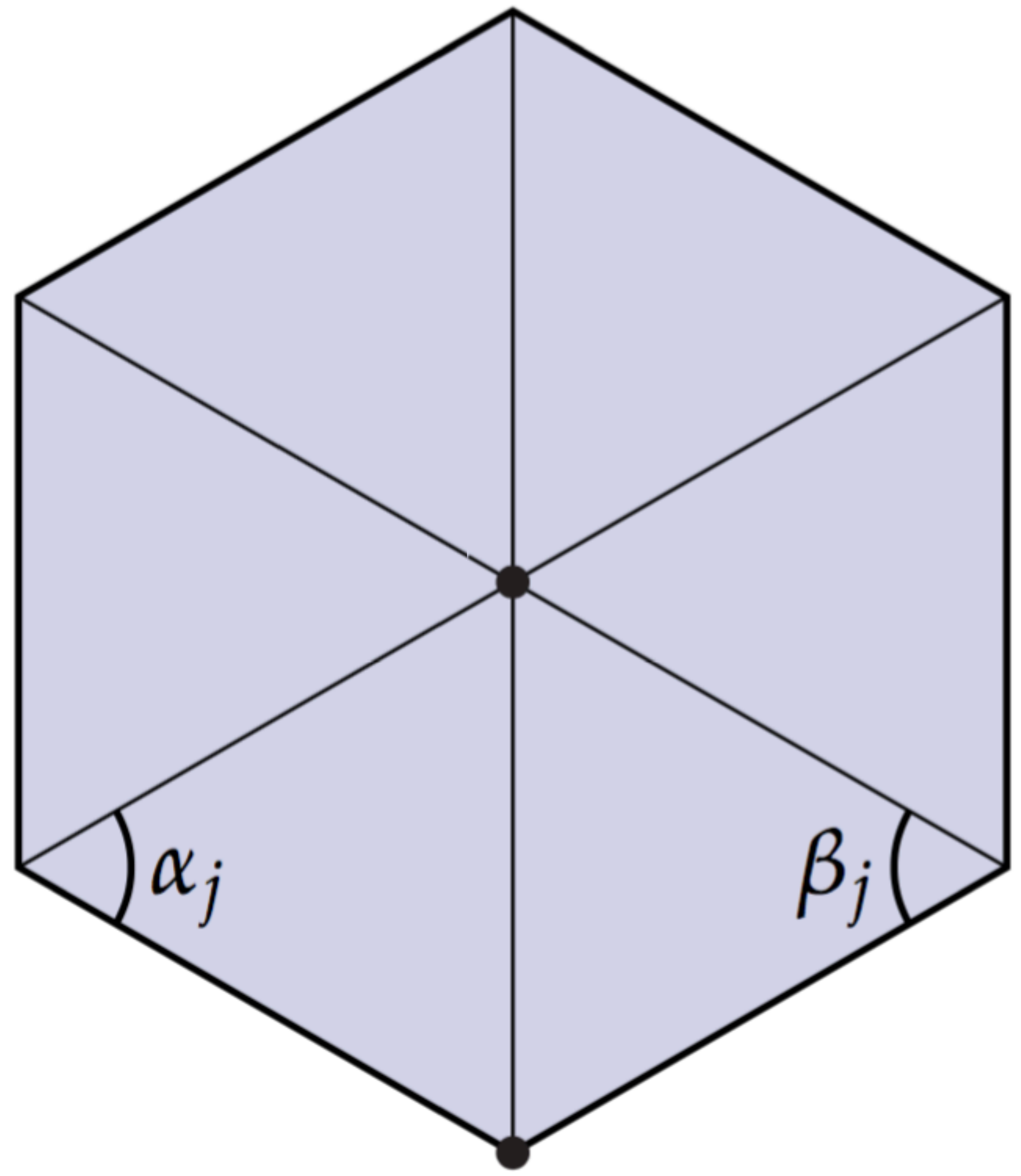
$$\Delta p_i = \frac{1}{2A_i} \sum_{j \in \mathcal{N}(i)} (\cot \alpha_{ij} + \cot \beta_{ij})(p_i - p_j)$$

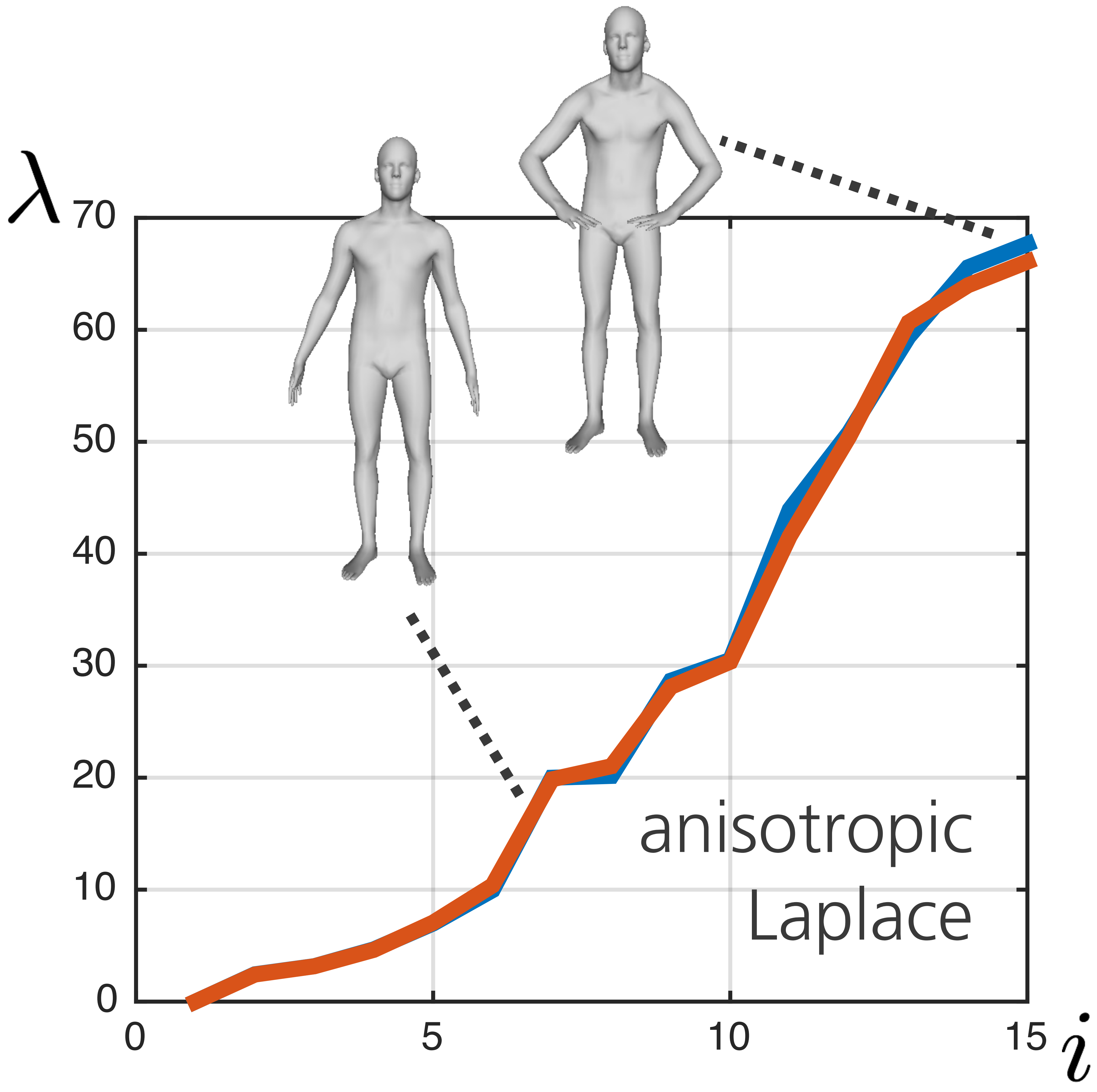
- Key idea:
  Laplace only depends on **edge lengths**!
  Edge length is a intrinsic quantity



Not Purely Intrinsic Operators

- Mixture of intrinsic and extrinsic information
- Existing operators:
  - Anisotropic Laplace
  - Modified Dirichlet energy

How **sensitive?**



Andreux et al. 2014, Hildebrandt et al. 2012

Quaternionic Dirac Operator D

- Definition:

$$D\psi = -\frac{df \wedge d\psi}{|df|^2}$$

Crane, et al. 2011

Quaternionic Dirac Operator D

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$$D\psi = -\frac{df \wedge d\psi}{|df|^2}$$

Quaternionic Dirac Operator D

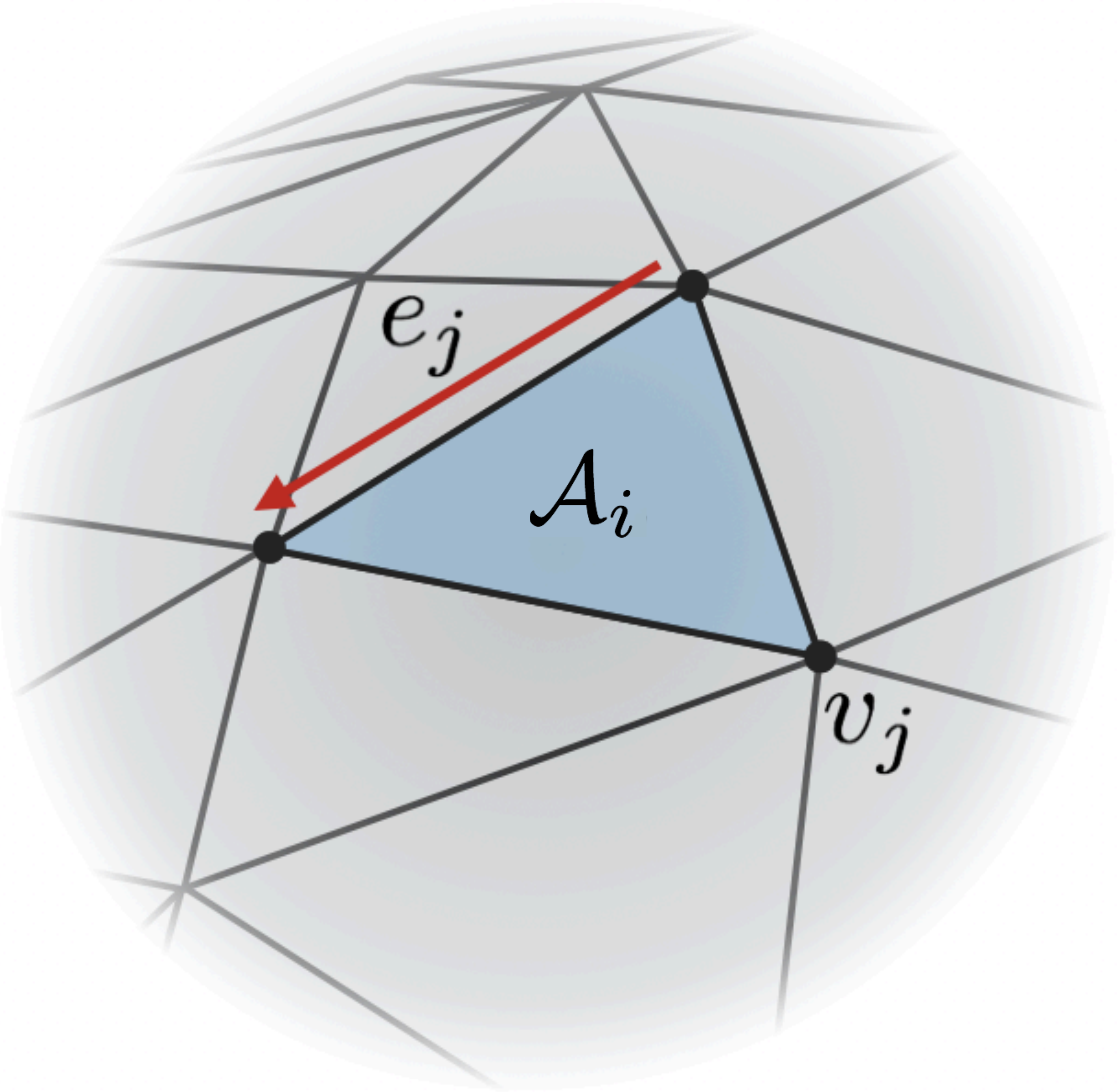
- Discrete Dirac:

$$\mathsf{D}_{ij} = \frac{-1}{2\mathcal{A}_i} e_j$$

- Key idea:

  depends on **edge vectors**

  (rather then edge length)



Crane, et al. 2011

Square of Dirac Operator

Laplace

Normal

$$D^2\psi = \Delta\psi + \frac{dN \wedge d\psi}{|df|^2}$$

Relative Dirac Operator D_N

intrinsic **extrinsic**

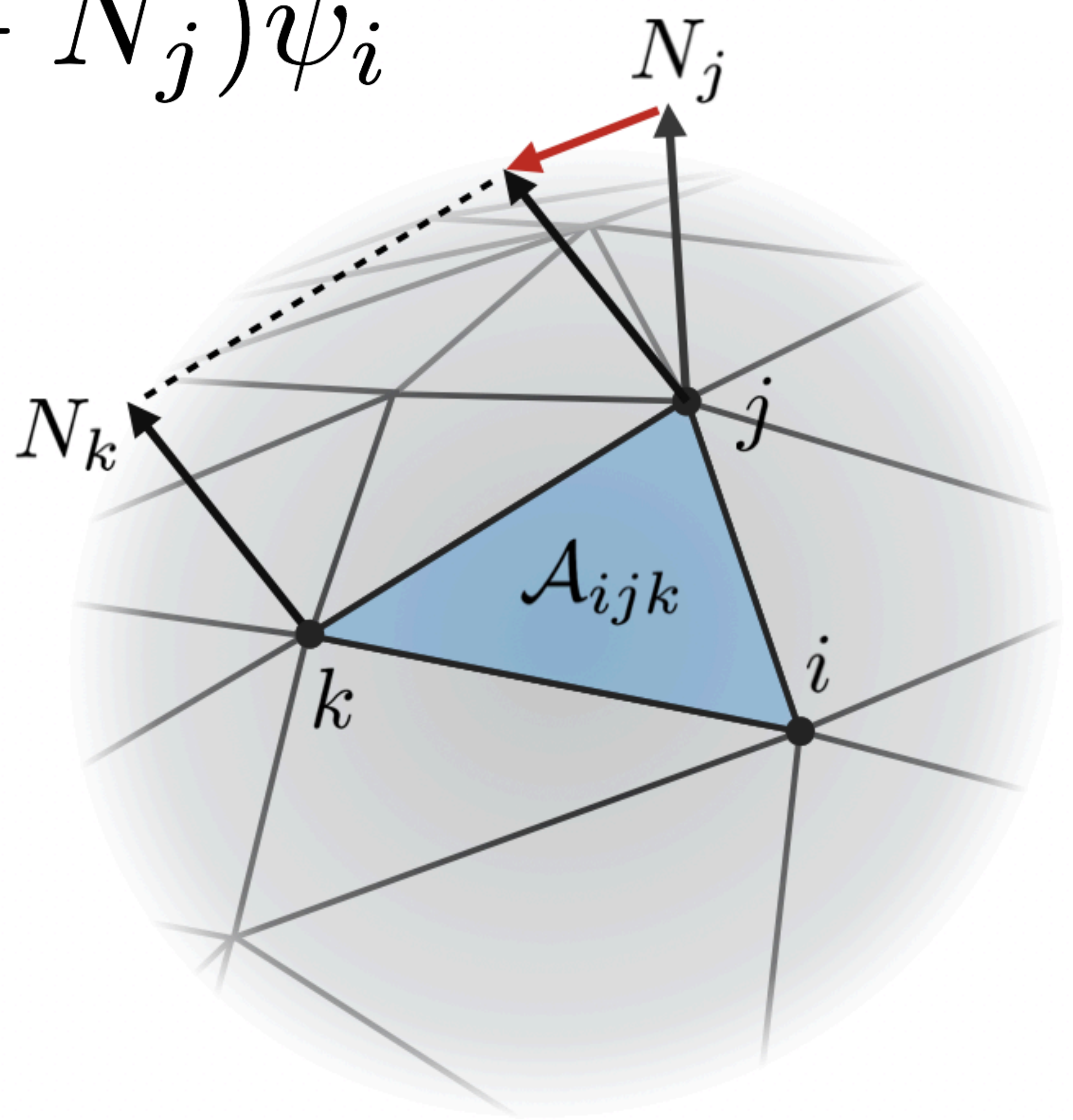
Discretization

- Discrete relative Dirac

$$(D_N\psi)_{ijk} = -\frac{1}{2\mathcal{A}_{ijk}} \sum_{pqr\in\mathcal{C}(ijk)} (N_k - N_j)\psi_i$$

- Matrix form

$$\mathrm{D_N}_{ijk,i} = -\frac{N_k - N_j}{2\mathcal{A}_{ijk}}$$



Basic Properties of Relative Dirac

(Dirac) (relative Dirac)

$$D\psi = -\frac{df \wedge d\psi}{|df|^2} \quad \longrightarrow \quad D_N\psi = -\frac{dN \wedge d\psi}{|df|^2}$$

Crane, et al. 2011

Basic Properties of Relative Dirac

(Dirac) (relative Dirac)

$$D\psi = -\frac{df \wedge d\psi}{|df|^2} \rightarrow D_N\psi = -\frac{dN \wedge d\psi}{|df|^2}$$

- First order, self-adjoint and elliptic operator
  → countable eigenvalues and eigenvectors

Crane, et al. 2011

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- The operator is not coordinate invariant
  → eigenvalues **are** coordinate invariant
  → eigenvectors **are** determined up to a constant

Crane, et al. 2011

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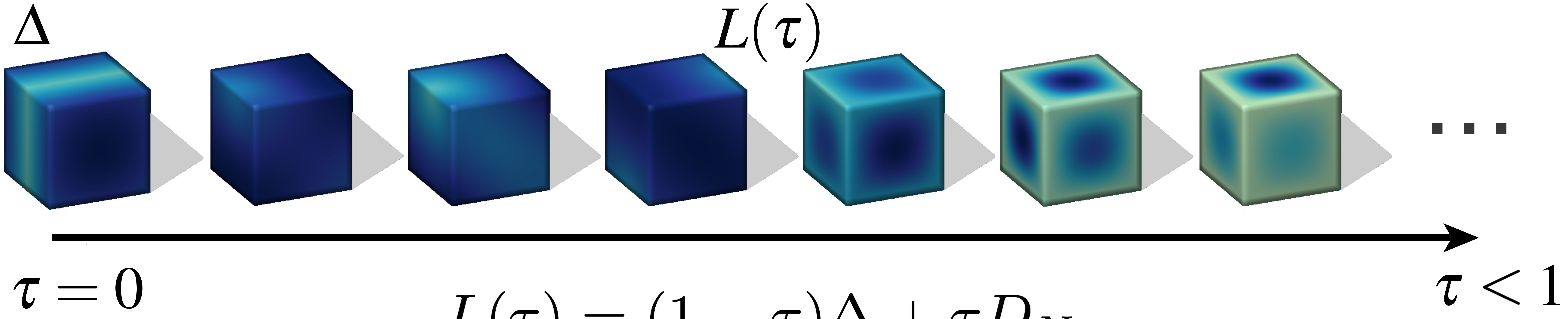
  → countable eigenvalues and eigenvectors
- The operator is not coordinate invariant
  → eigenvalues **are** coordinate in...
  → eigenvectors **are** de... o a constant

Not isometry invariant!!

Crane, et al. 2011

From Intrinsic to Extrinsic

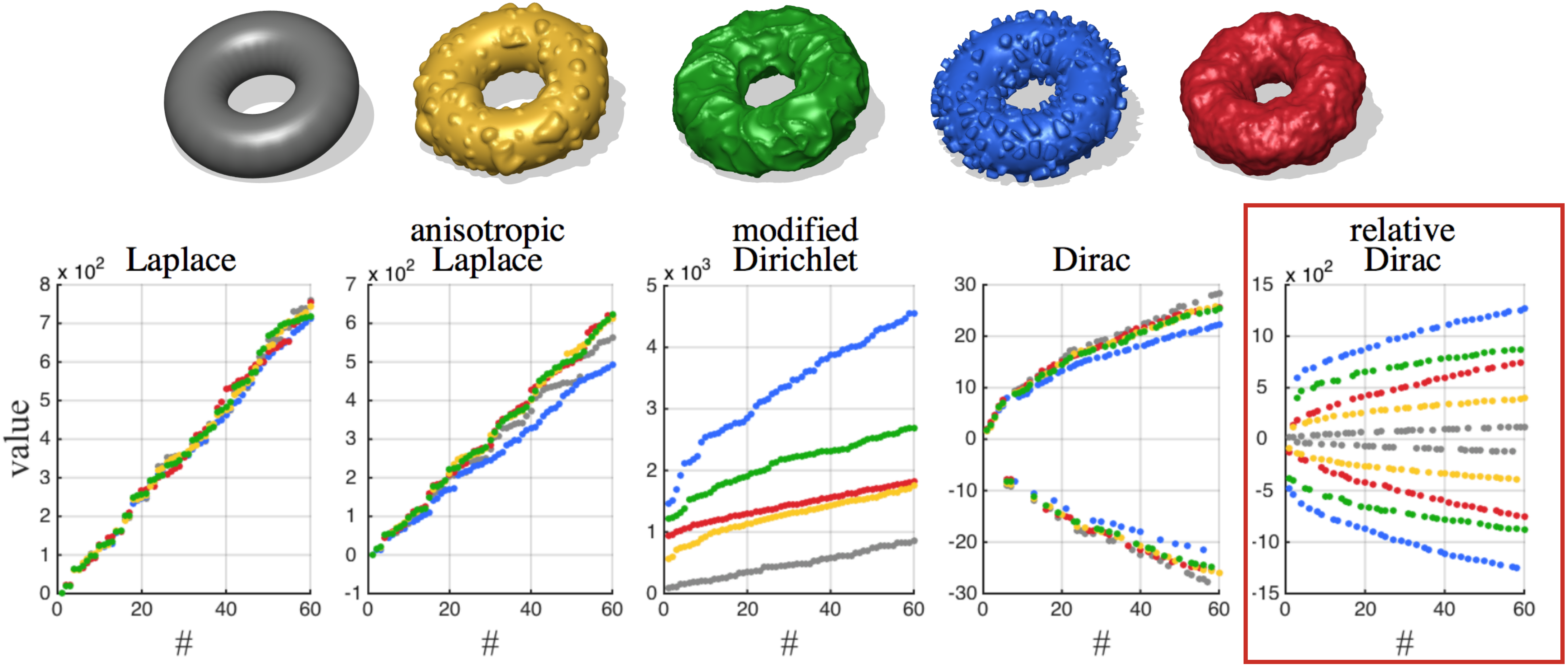


$$L(\tau) = (1 - \tau)\Delta + \tau D_N$$

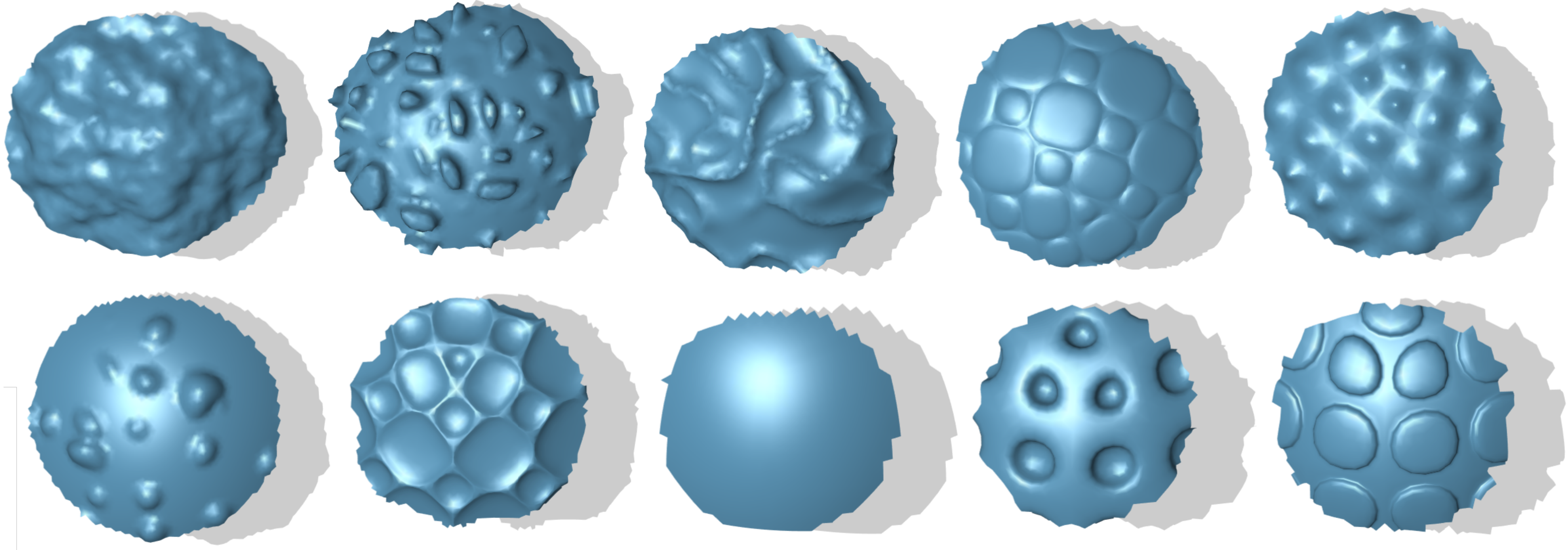
Laplace   relative Dirac

Applications

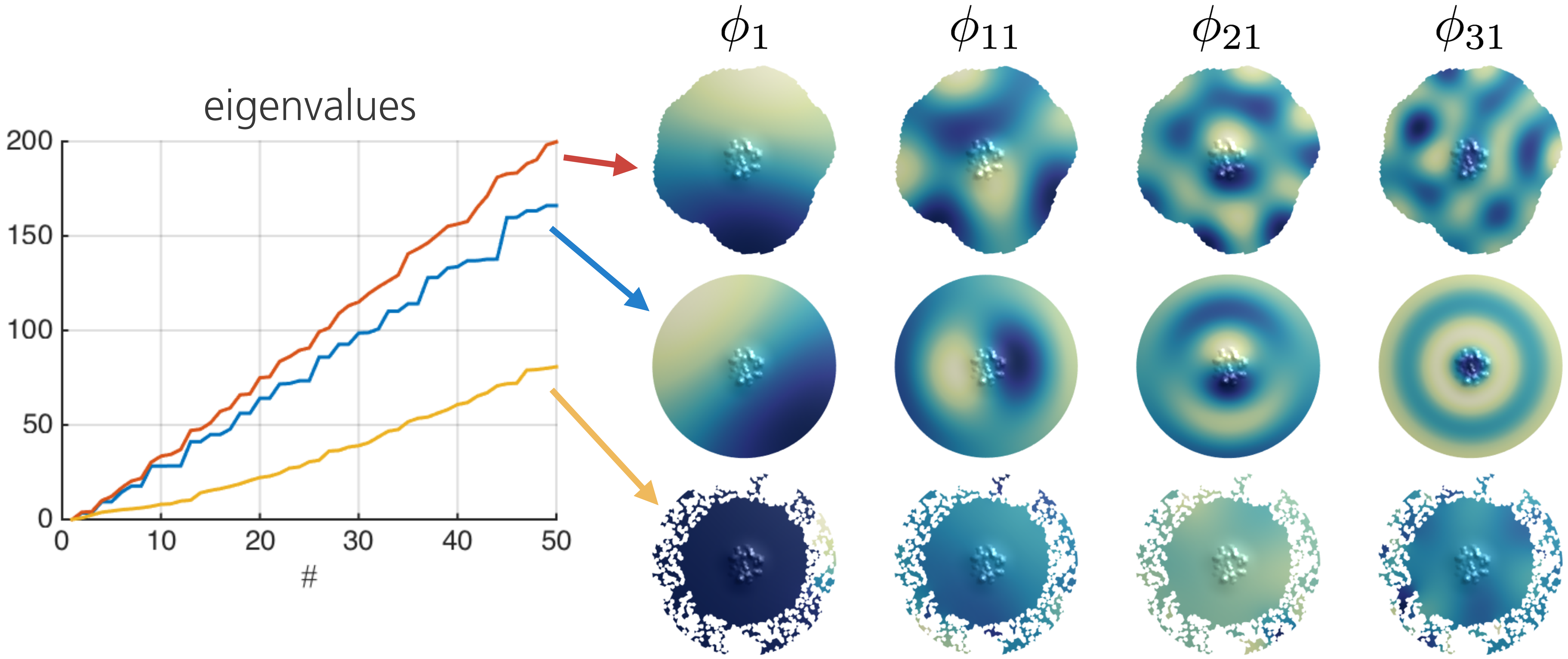
Surface Texture Classification



Patch Classification



Laplace



eigenvalues
200
150
100
50
0
0
10
20
30
40
50
#
ϕ_1
ϕ_{11}
ϕ_{21}
ϕ_{31}

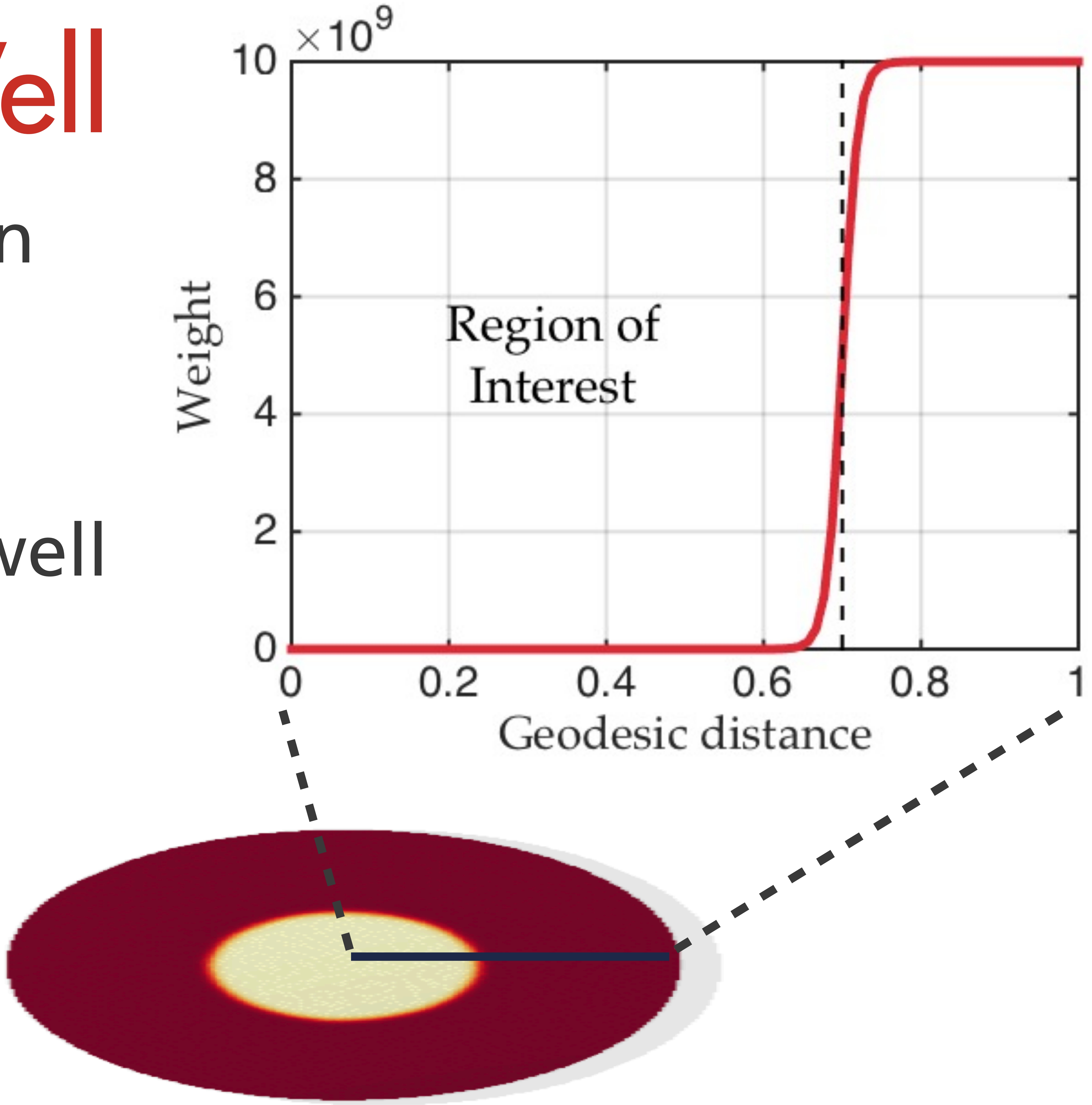
Infinite Potential Well

- Modified sigmoid function

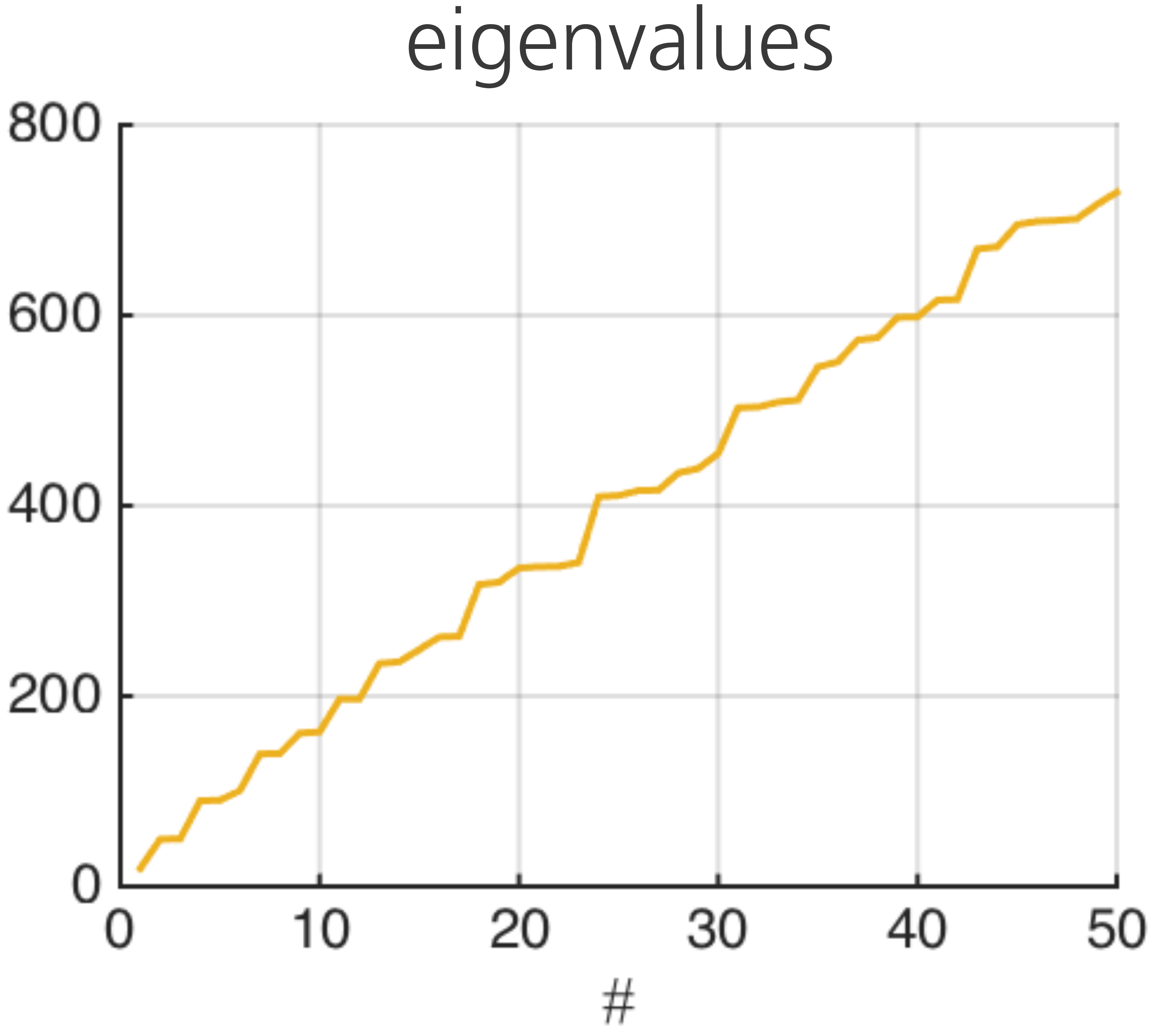
$$U(p) = \frac{c}{1 + (e^{-(d(p,q)-\beta)})^{\gamma}}$$

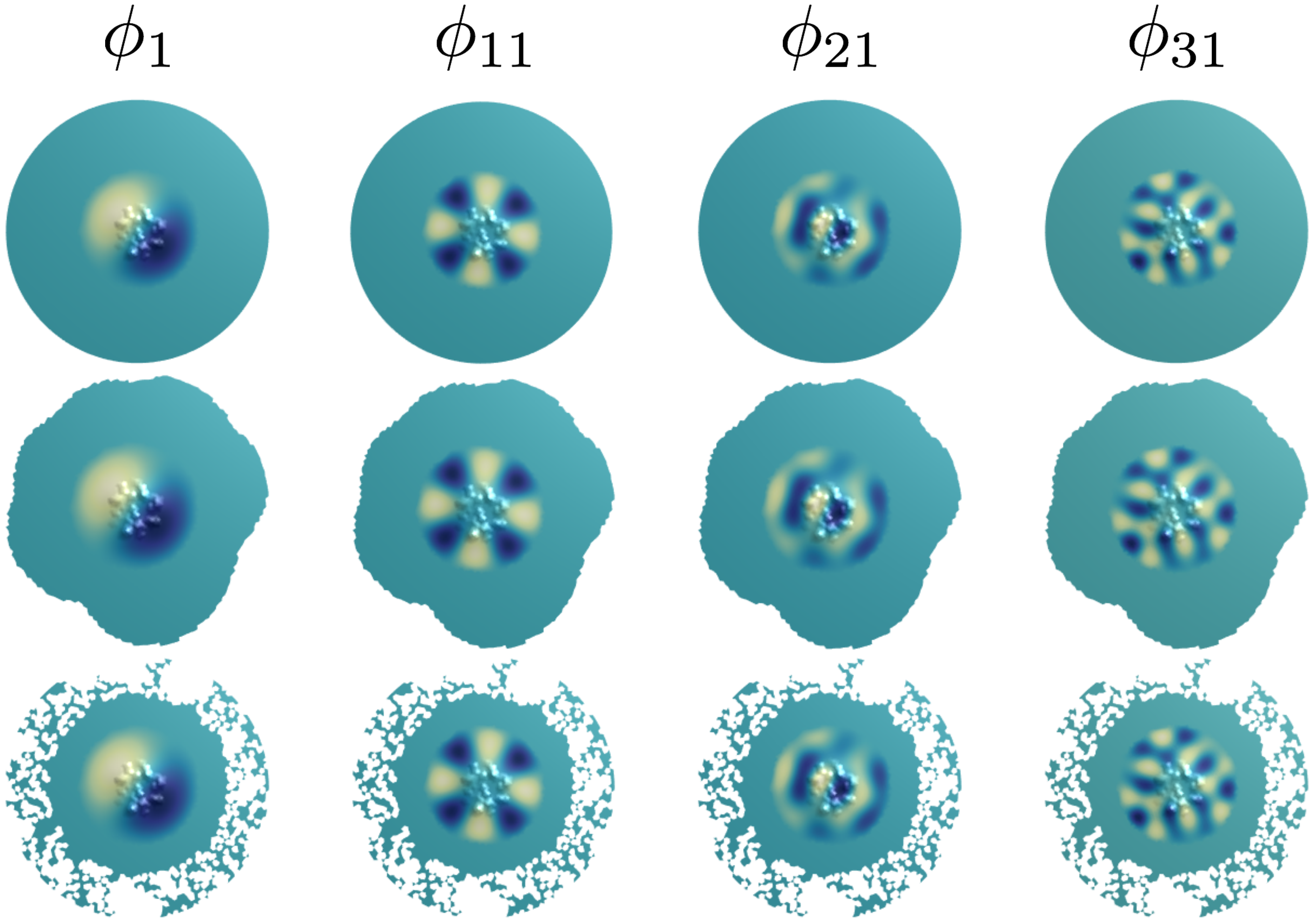
- Operator with potential well

Ex: $\Delta \rightarrow \Delta + U$



Laplacian with Infinite Potential Well





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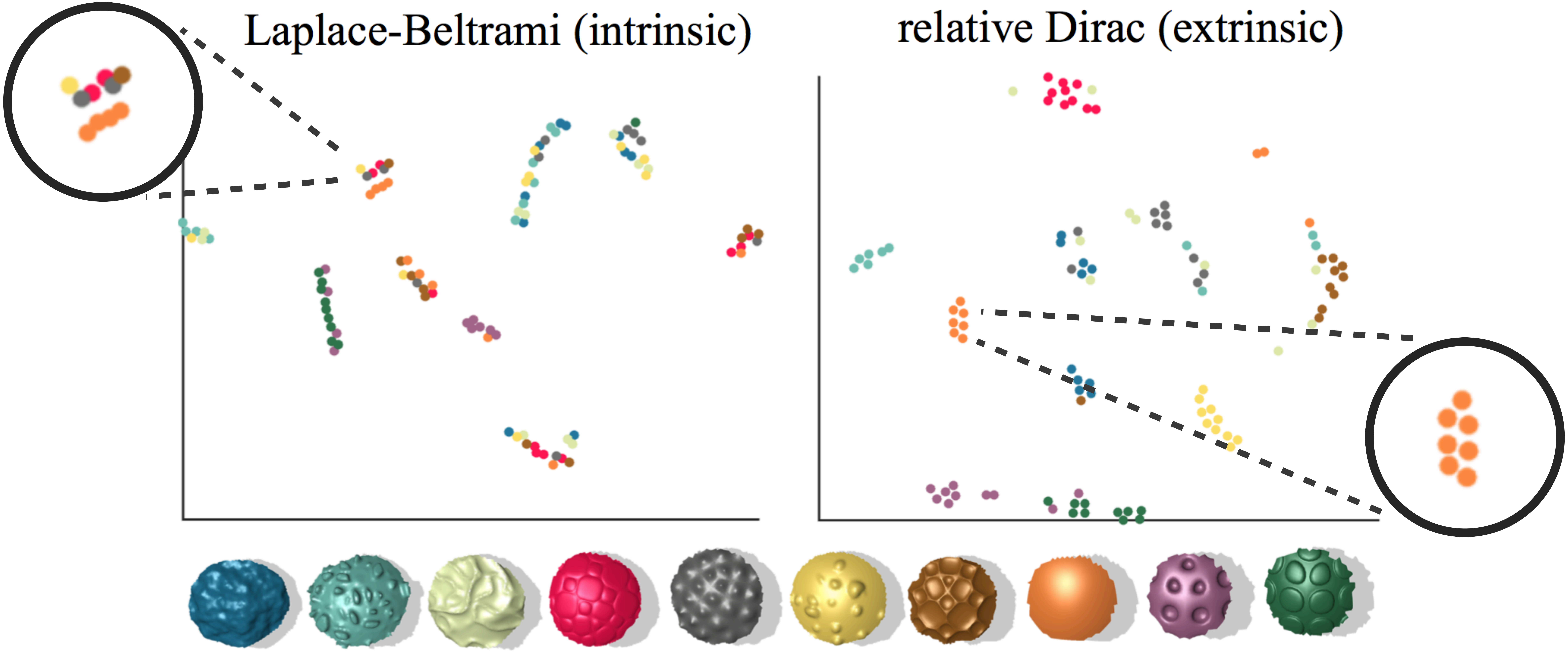


Maaten, et al. 2008

Segmentation

Step 1: Adapt **global point signature** to the magnitude of Dirac

eigenvectors

$$v \mapsto \left(\frac{|\phi_1(v)|}{\sqrt{\lambda_1}}, \frac{|\phi_2(v)|}{\sqrt{\lambda_2}}, \frac{|\phi_3(v)|}{\sqrt{\lambda_3}}, \ldots \right)$$

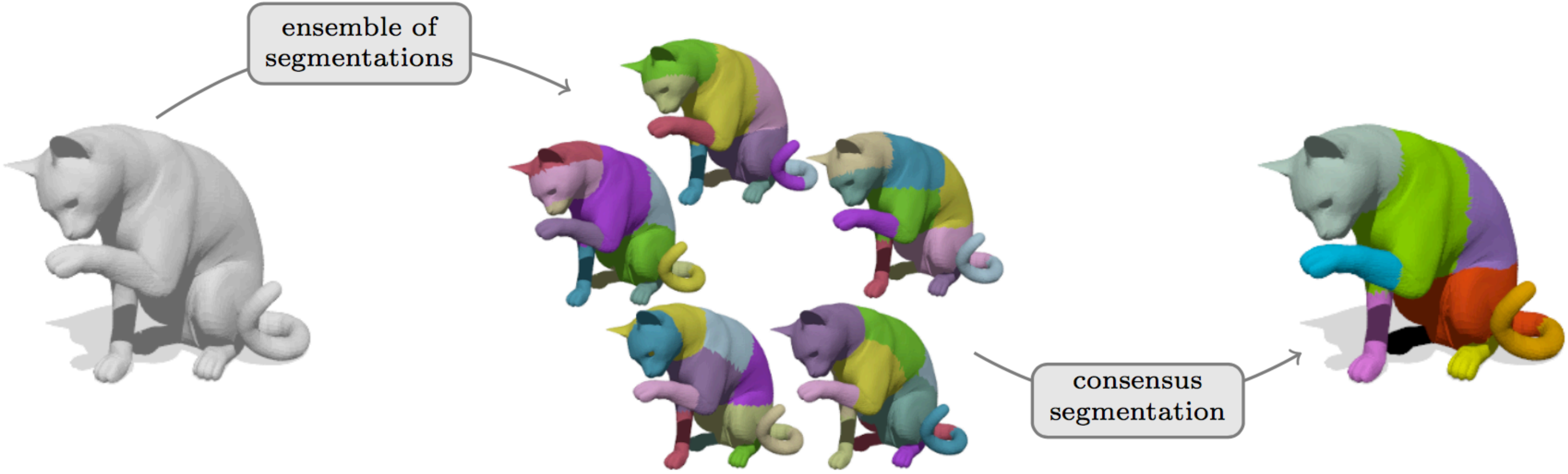
point on the surface

eigenvalues

Rustamov, et al. 2007

Segmentation

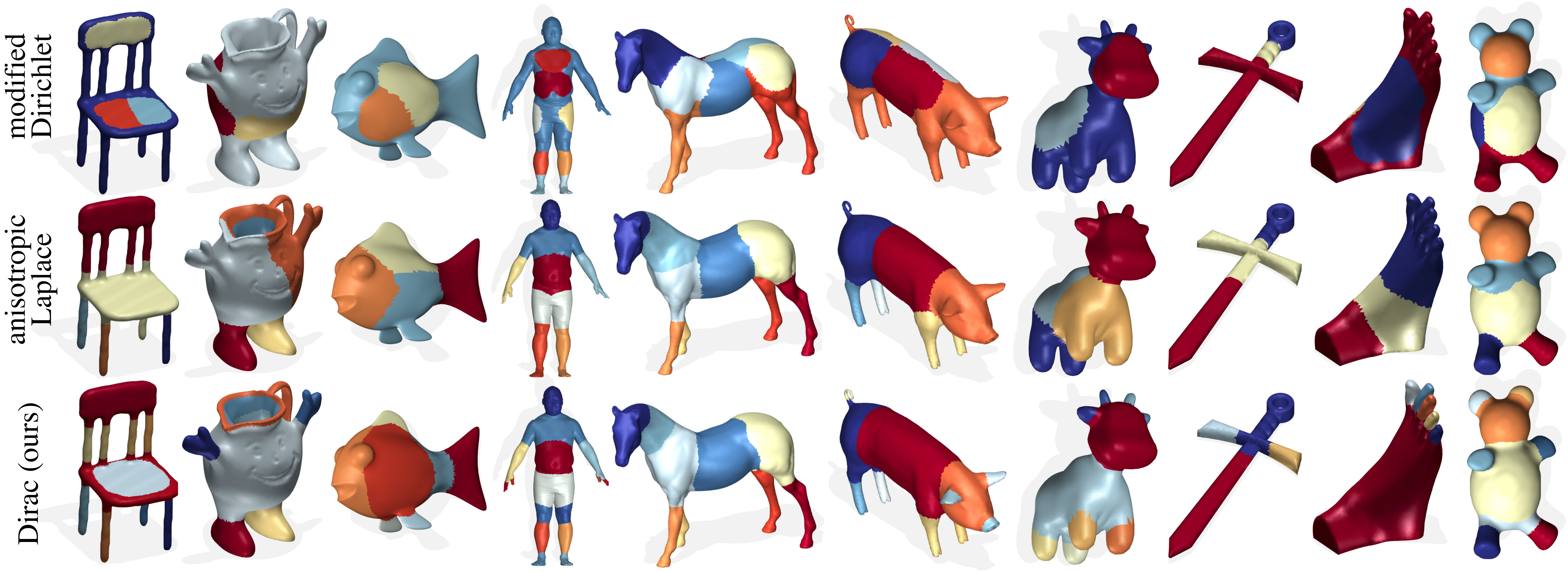
Step 2: Apply the **consensus segmentation** algorithm



random initialized k-means

Rodola, et al. 2014

Segmentation



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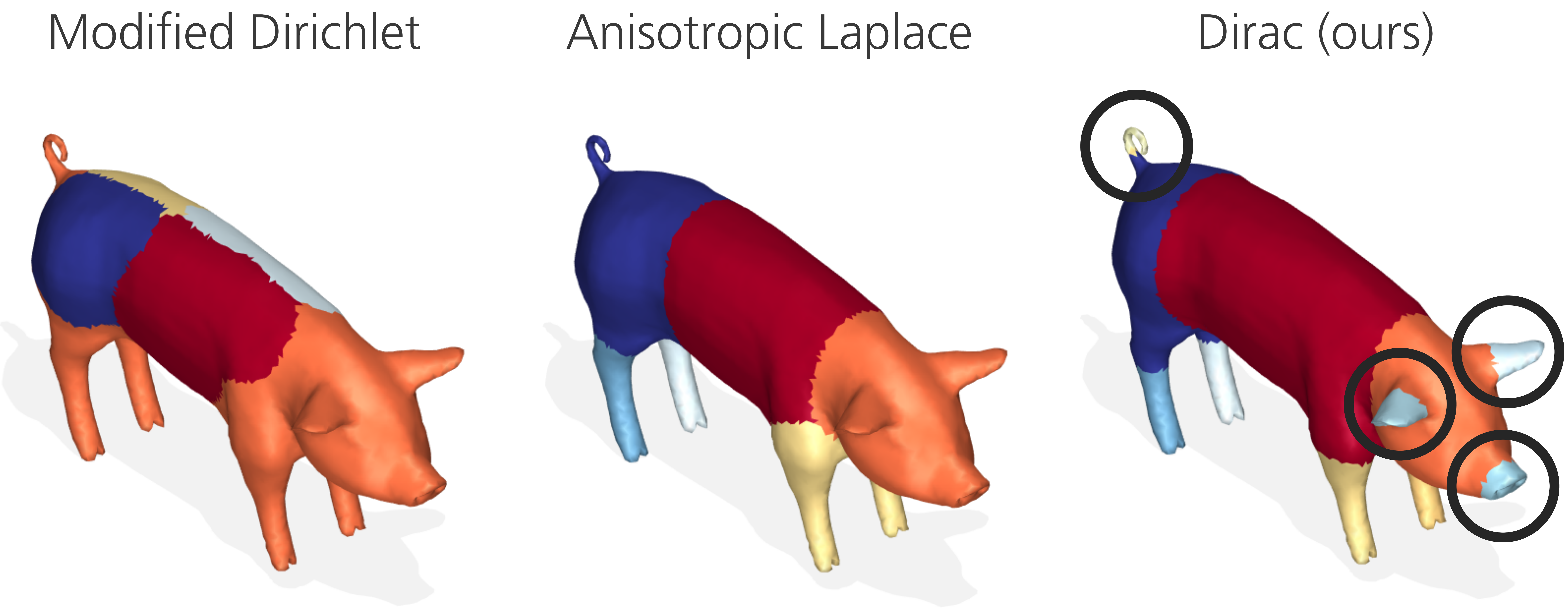
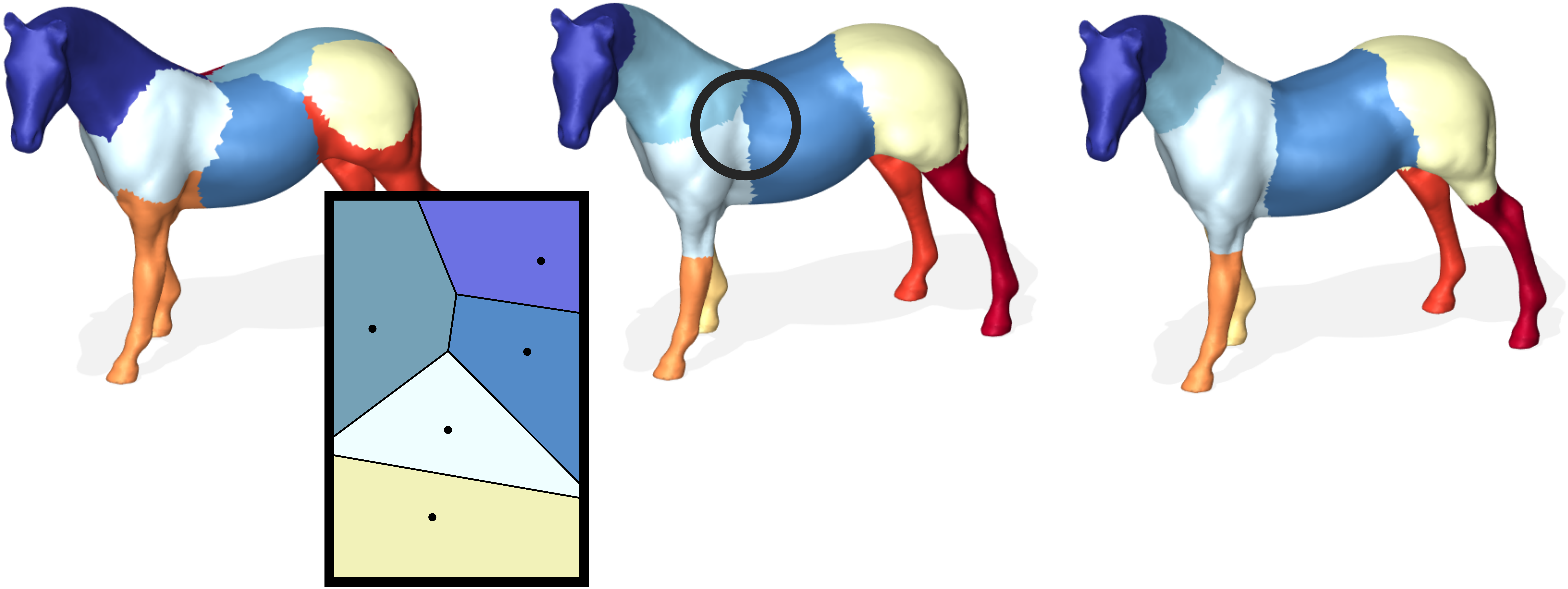


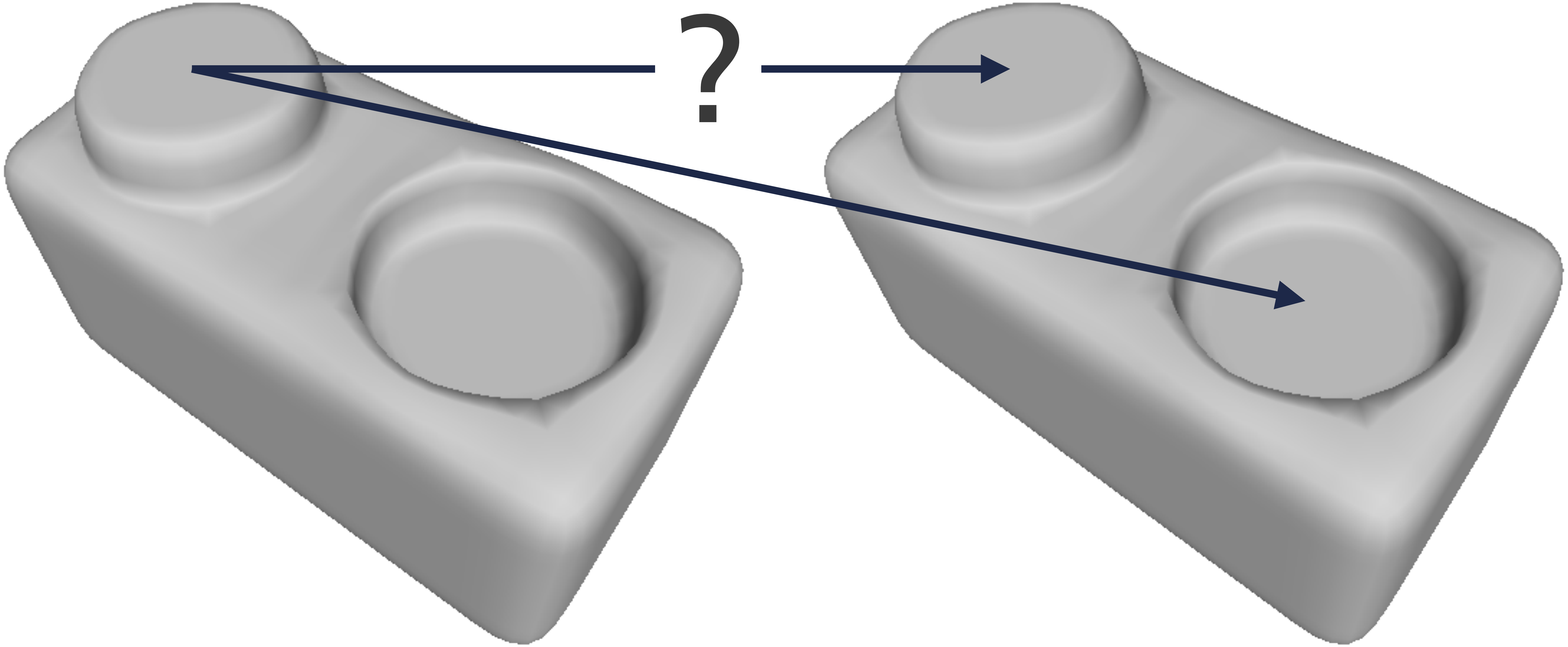
Segmentation

Modified Dirichlet | Anisotropic Laplace | Dirac (ours)

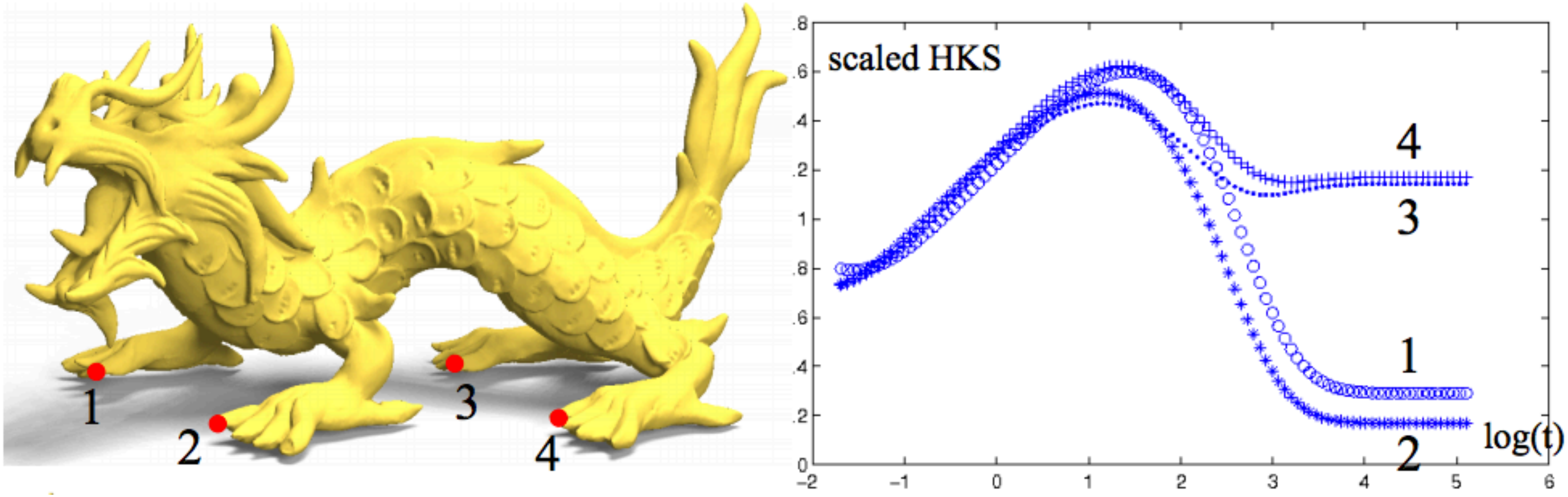


Correspondence

- Laplace cannot differentiate between bumped out/in



Heat Kernel Signature



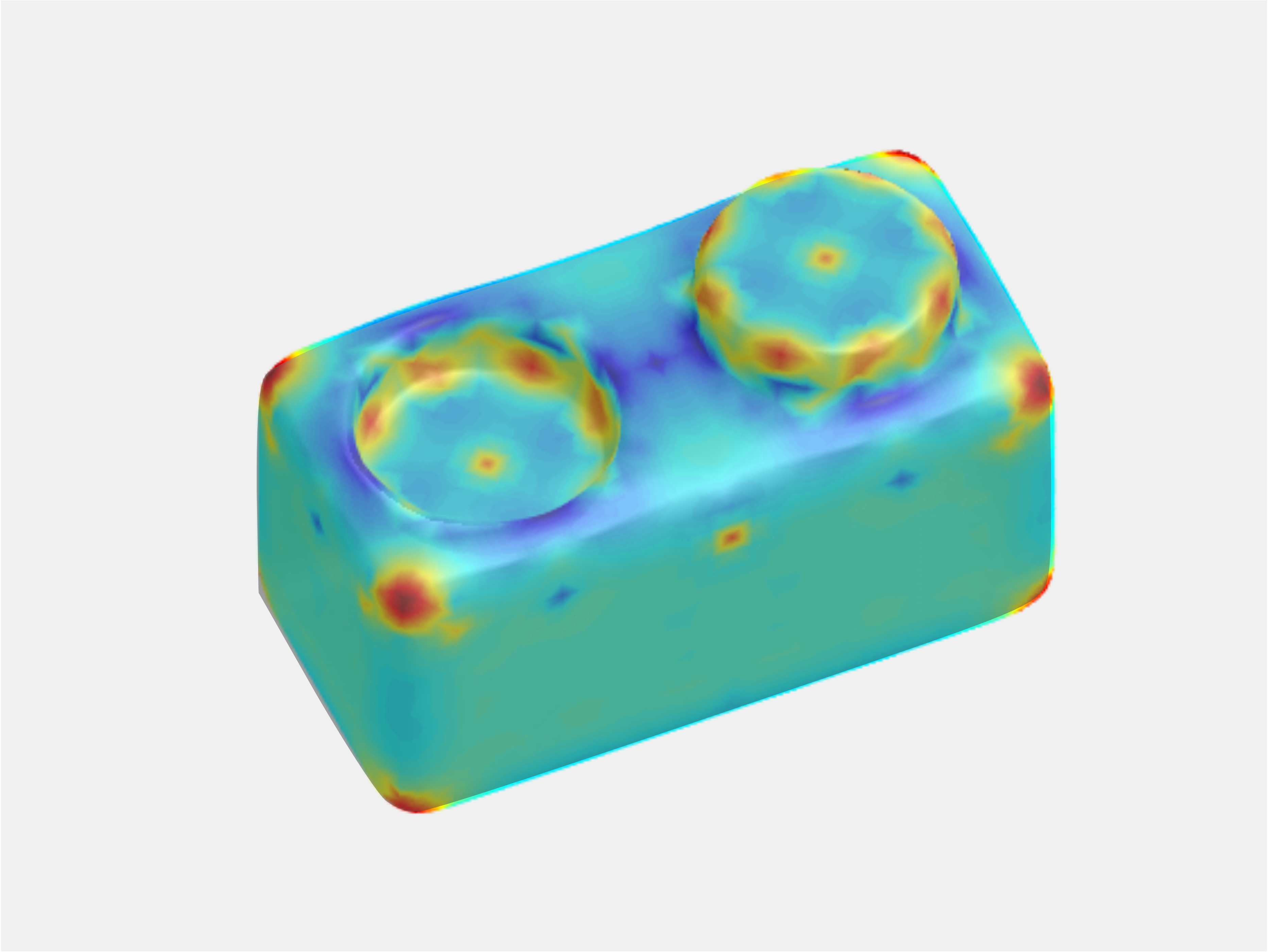
Sun, et al. 2009

Correspondence

- Adapt heat kernel signature to the Dirac operator

Dirac kernel signature $\mathbb{H}$

heat kernel signature $\mathbb{R}$



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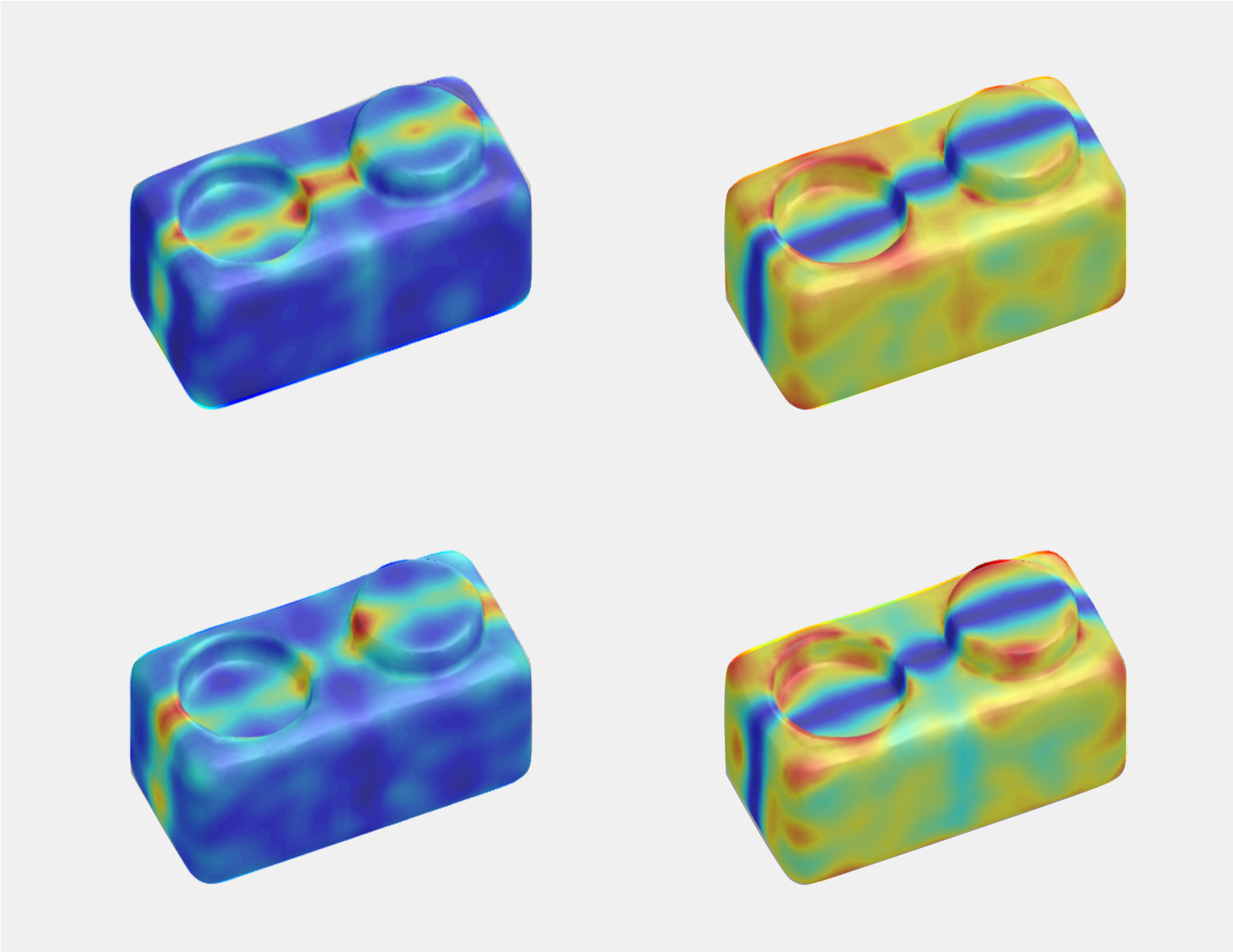

heat kernel signature $\mathbb{R}$

Correspondence

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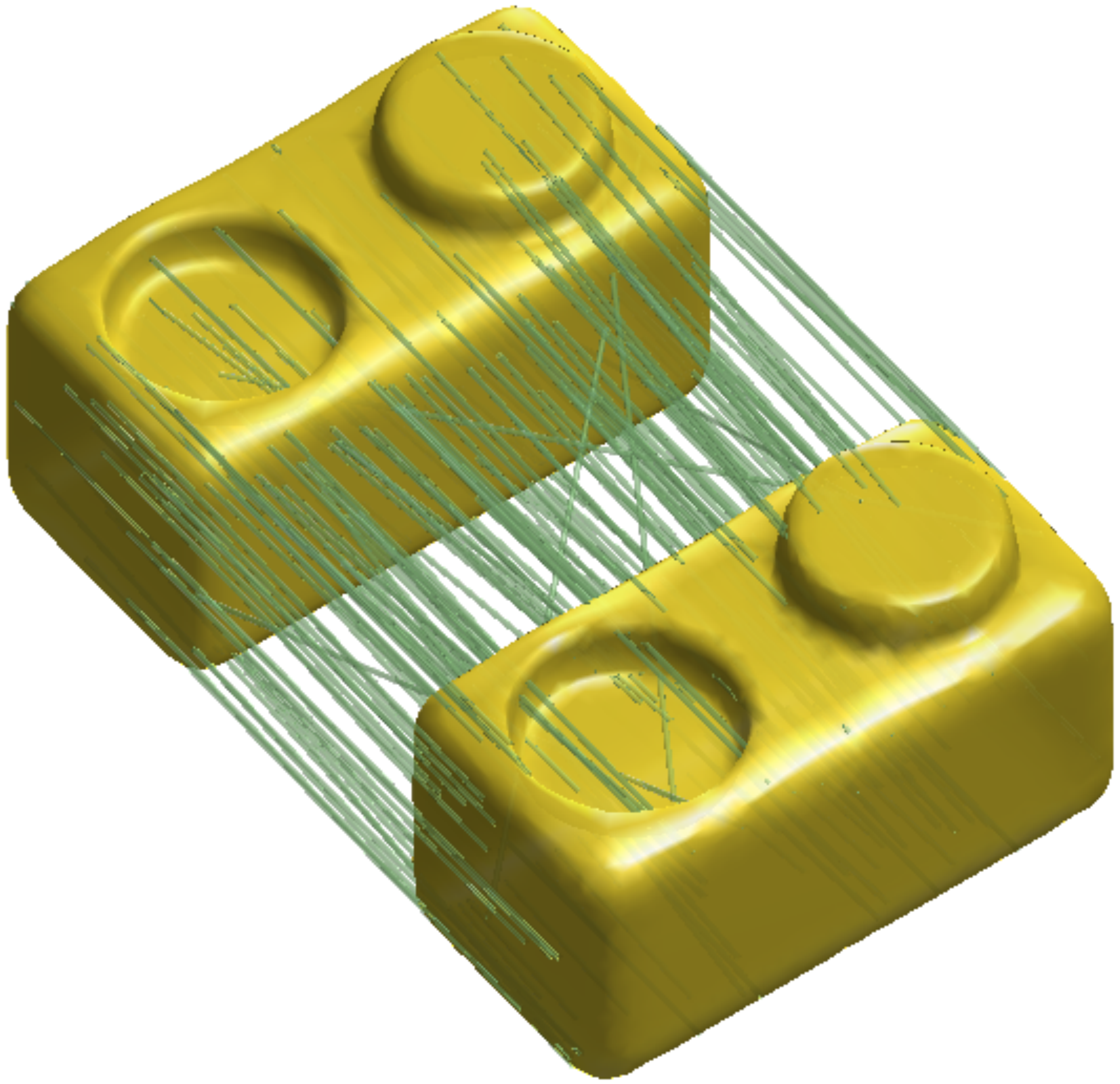

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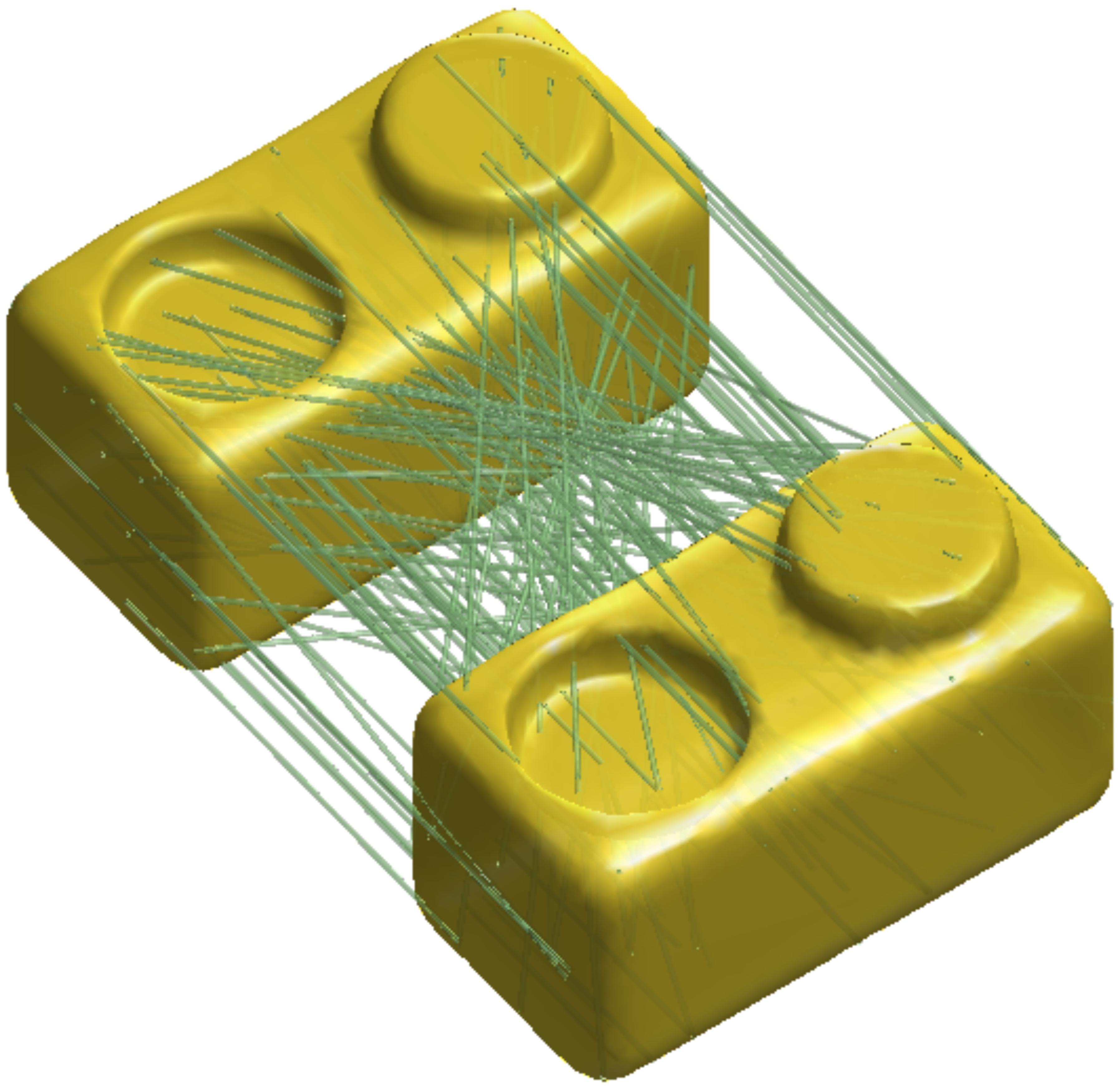

Conclusion

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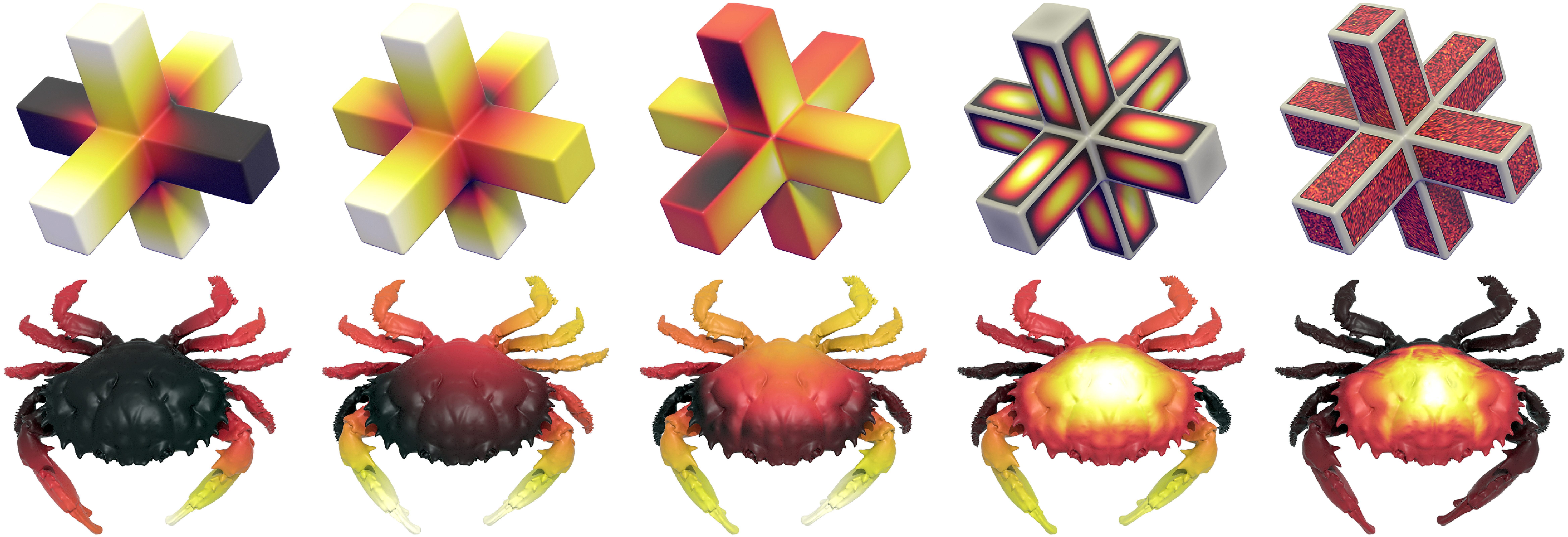
https://github.com/alecjacobson/gptoolbox.git



Thank you!

Hsueh-Ti Derek Liu, hsuehtil@andrew.cmu.edu