Tutorial #9 – Parametric curves
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Possible representation of curves: explicit, implicit and parametric

Explicit representation
- curve in 2D y=f(x)
- curve in 3D y=f(x), z=g(x)

- easy to compute a point (given parameters)
- multiple values of y and z for a single x is impossible (e.g., circle where $y = \pm\sqrt{r^2 - x^2}$ has 2 or 0 values)
- how to represent an infinite slope?

Implicit representation

- curve in 2D: F(x,y)=0
    - line: ax+by+c=0
    - circle: $x^2 + y^2 - r^2 = 0$
- surface in 3D: F(x,y,z)=0
    - plane: ax+by+cz+d=0

- F(x,y,z)=0 describes a 3D object $\begin{cases} F(x,y,z) < 0, \text{inside} \\ F(x,y,z) = 0, \text{on surface} \\ F(x,y,z) > 0, \text{inside} \end{cases}$

- However, computing a point is difficult (it is easy to determine where any point lies in terms of that curve).

Parametric representation

- curves: single parameter (e.g $u \in [0,1]$)
    - x=X(u), y=Y(u), z=Z(u)
- tangent is simply derivatives of the above functions

- Advantages:
    - capable of precisely representing any curve a user may need
    - easy and efficient to represent in a computer
    - little memory storage required

Parametric cubic curves

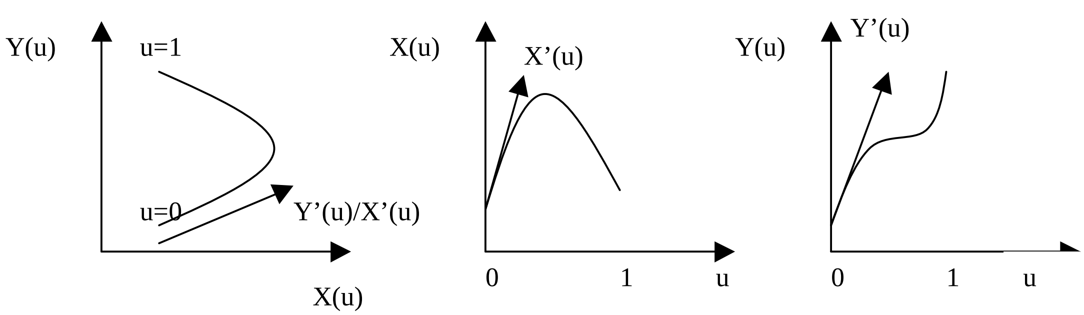
- Why cubic?
- Curves of lower order commonly have too little flexibility
- Curves of higher order are unnecessarily complex and can introduce certain artifacts
- Cubic curves offer a good trade-off between complexity and flexibility

$$\begin{cases} X(u) = a_3u^3 + a_2u^2 + a_1u + a_0 \\ Y(u) = b_3u^3 + b_2u^2 + b_1u + b_0 \\ Z(u) = c_3u^3 + c_2u^2 + c_1u + c_0 \end{cases}$$

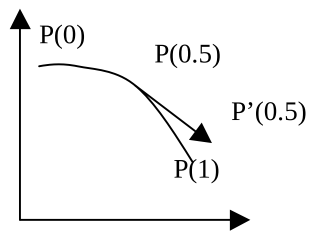
- compact representation

$$X(u) = [u^3, u^2, u \ \ ,1]\begin{bmatrix} a_3 \\ a_2 \\ a_1 \\ a_0 \end{bmatrix} \qquad \text{i.e.,} \quad \begin{matrix} X(u) = UA \\ Y(u) = UB \\ Z(u) = UC \end{matrix}$$

- derivatives are easy: $\frac{dX(u)}{du} = \begin{bmatrix} 3u^2 & 2u & 1 & 0 \end{bmatrix} A$
- think of u as a time parameter



Find the curve parameters given that the user specifies the following 4 parameters:
- 2 end points
- midpoint
- tangent at midpoint



$$\begin{bmatrix} X(0) \\ X(0.5) \\ X'(0.5) \\ X(1) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0.5^3 & 0.5^2 & 0.5 & 1 \\ 3(0.5)^2 & 2(0.5) & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} A$$

- $G_x = BA \Rightarrow A = B^{-1}G_x$

- since $X(u)=UA$, we have $X(u) = UB^{-1}G_x \Rightarrow X(u) = UMG_x$, where M is called the basis matrix, which in this case is

$$M = \begin{bmatrix} -4 & 0 & -4 & 4 \\ 8 & -4 & 6 & -4 \\ -5 & 4 & -2 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

- we can then rewrite our equation

$$X(u) = UMG_x = [f_1(u), \quad f_2(u), \quad f_3(u), \quad f_4(u)]G_x,$$

where

$$f_1(u) = -4u^3 + 8u^2 - 5u + 1$$
$$f_2(u) = -4u^2 + 4u$$
$$f_3(u) = -4u^3 + 6u^2 - 2u$$
$$f_4(u) = 4u^3 - 4u^2 + 1$$

(similarly for Y and Z).

Parametric Continuity (for different curves p and q)

- C_0: matching endpoints p(1)=q(0)
- C_1: matching derivatives (with same magnitude) p'(1)=q'(0)
- C_2: 1st and 2nd derivatives are equal
- C_n: nth derivatives are equal

Geometric continuity

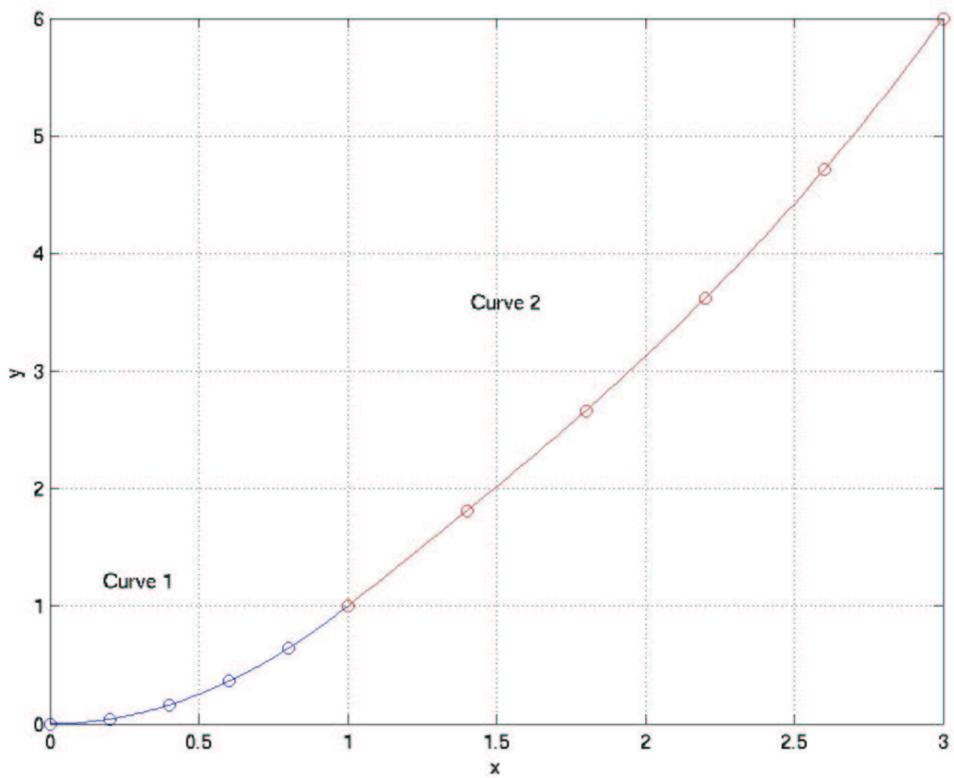
- $G_0=C_0$
- G_1: matching derivatives (not necessarily magnitude)
- G_2: 1st and 2nd derivatives proportional
- G_n: nth derivatives are proportional

Example: Determine if the following curves are C_0, C_1, G_1 continuous

$$A(u) = (u, u^2)$$
$$B(u) = (2u+1, u^3 + 4u + 1) \quad \text{for } u \in [0,1]$$

A(1)=(1,1) and B(0)=(1,1), thus they're C_0 continuous

A'(1)=(1,2) and B'(0)=(2,4), magnitudes are different but derivatives are proportional, thus they're G_1 continuous, but not C_1.



Note the difference in speed between curves 1 and 2.