Tutorial 10. Parametric Curves (continued)

Review of three cubic curves.

- First representation in notes. Two end-points, mid-point, tangent vector at midpoint.
- Hermite Curves. Two end-points, two tangent vectors at start and end. Offer C1 continuity.
- Bezier Curves. Two end-points, two control points. Offer C1 continuity.

Today's topic: A few examples on other kinds of cubic curves.

Example 1. Sketch a parametric cubic curve defined by the following 4 points, and give the basis matrix, basis functions, and a sketch of the basis functions.
$P(0) = P_1, P(\frac{1}{3}) = P_2, P(\frac{2}{3}) = P_3, P(1) = P_4$.

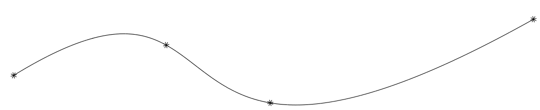

Figure 1: A cubic curve and its $P(0), P(\frac{1}{3}), P(\frac{2}{3}), P(1)$.

We will solve for X; Y and Z can be solved similarly.

$$X(t) = TA, \qquad \text{where} \quad T = [t^3, t^2, t, 1], A = [a_3, a_2, a_1, a_0]^T$$

Here are our 4 constraints:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ (\frac{1}{3})^3 & (\frac{1}{3})^2 & \frac{1}{3} & 1 \\ (\frac{2}{3})^3 & (\frac{2}{3})^2 & \frac{2}{3} & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a_3 \\ a_2 \\ a_1 \\ a_0 \end{pmatrix}$$

$$\text{which is:} G = BA$$

$$A = B^{-1}G$$

$$X(t) = TA = TB^{-1}G = TMG$$

$$M = \frac{1}{2}\begin{pmatrix} -9 & 27 & -27 & 9 \\ 18 & -45 & 36 & -9 \\ -11 & 18 & -9 & 2 \\ 2 & 0 & 0 & 0 \end{pmatrix}$$

Basis functions, TM:

$$\begin{aligned} f_1(t) &= -4.5t^3 + 9t - 5.5t + 1 \\ f_2(t) &= 13.5t^3 - 22.5t^2 + 9t \\ f_3(t) &= -13.5t^3 + 18t^2 - 4.5t \\ f_4(t) &= 4.5t^3 - 4.5t^2 + t \end{aligned}$$

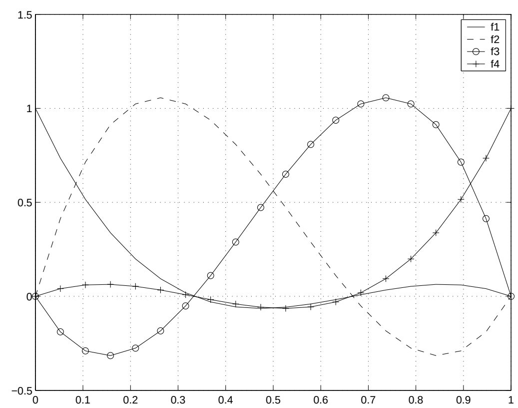

Figure 2: Plot of the basis functions.

Example 2. Suppose we want to define a C2 continuous curve to connect two curves.

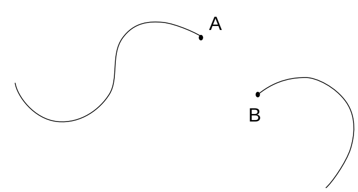

Figure 3: Two cubic curves to be connected at A, B.

If A=B, there is nothing we can do. Otherwise, suppose the new curve is of the form:

$$X(t) = a_3t^3 + a_2t^2 + a_1t + a_0$$

We have 4 variables a_3, a_2, a_1, a_0, and 6 constraints: $x(0)$, $x(1)$, $x'(0)$, $x'(1)$, $x''(0)$, $x''(1)$, in general, there is no solution. What about connecting the two curves with guarantee of C1 continuity?

Example 3. Splines are cubic curves that maintain C2 continuity. B-Splines are Splines that offer local control:

$$Q_i(t) = TM_{BS}G_{BS,i}$$

where $T = [t^3, t^2, t, 1]$, $M_{BS} = \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{pmatrix}$, $G_{BS,i} = \begin{pmatrix} P_{i-3} \\ P_{i-2} \\ P_{i-1} \\ P_i \end{pmatrix}$, $i \in [3 \dots m]$

It has $m+1$ control points, $m-2$ curve segments.

Now, let's verify that two adjacent curve segments i and $i{+}1$ are C2 continuous.

$$X_i(1) = X_{i+1}(0) \Longleftarrow \begin{cases} X_i(1) = [1,1,1,1]M_{BS}\begin{pmatrix} x_{i-3} \\ x_{i-2} \\ x_{i-1} \\ x_i \end{pmatrix} = [0,1,4,1]\begin{pmatrix} x_{i-3} \\ x_{i-2} \\ x_{i-1} \\ x_i \end{pmatrix} \\ X_{i+1}(0) = [0,0,0,1]M_{BS}\begin{pmatrix} x_{i-2} \\ x_{i-1} \\ x_i \\ x_{i+1} \end{pmatrix} = [1,4,1,0]\begin{pmatrix} x_{i-2} \\ x_{i-1} \\ x_i \\ x_{i+1} \end{pmatrix} \end{cases}$$

$$X_i'(1) = X_{i+1}'(0) \Longleftarrow \begin{cases} X_i'(1) = [3,2,1,0]M_{BS}\begin{pmatrix} x_{i-3} \\ x_{i-2} \\ x_{i-1} \\ x_i \end{pmatrix} = [0,-3,0,3]\begin{pmatrix} x_{i-3} \\ x_{i-2} \\ x_{i-1} \\ x_i \end{pmatrix} \\ X_{i+1}'(0) = [0,0,1,0]M_{BS}\begin{pmatrix} x_{i-2} \\ x_{i-1} \\ x_i \\ x_{i+1} \end{pmatrix} = [-3,0,3,0]\begin{pmatrix} x_{i-2} \\ x_{i-1} \\ x_i \\ x_{i+1} \end{pmatrix} \end{cases}$$

$$X_i''(1) = X_{i+1}''(0) \Longleftarrow \begin{cases} X_i''(1) = [6,2,0,0]M_{BS}\begin{pmatrix} x_{i-3} \\ x_{i-2} \\ x_{i-1} \\ x_i \end{pmatrix} = [0,6,-12,6]\begin{pmatrix} x_{i-3} \\ x_{i-2} \\ x_{i-1} \\ x_i \end{pmatrix} \\ X_{i+1}''(0) = [0,2,0,0]M_{BS}\begin{pmatrix} x_{i-2} \\ x_{i-1} \\ x_i \\ x_{i+1} \end{pmatrix} = [6,-12,6,0]\begin{pmatrix} x_{i-2} \\ x_{i-1} \\ x_i \\ x_{i+1} \end{pmatrix} \end{cases}$$