Fluid Simulation on S^2 using Spherical Harmonics

Michael Tao

September 19, 2011

1 Introduction

This is a report for my CSC495 project course taken under Professor Eugene Fiume on the implementation of Tyler De Witt's fluid simulation method for velocity fields representing fluid flow on the surface of a sphere using spherical harmonics.

The method models incompressible fluids governed by the Navier Stokes by working in the vorticity space by using harmonic basis functions over the given domain for the flow (in this case the surface of a sphere). Vorticity measures the spin of a vector field, and is represented by the curl of the vector field, a rank 1 map from $S^2 \to \mathbb{R}^3$ normal to the velocity space $\subset \mathcal{T}S^2$, the tangent space for S^2. This means that the vorticity must always be normal to the surface of the sphere. The vorticity is modeled by the spherical harmonics, which are an orthogonal family of basis functions for finite energy functions on $S^2 \to \mathbb{R}$, by attaching an $\hat{\boldsymbol{r}}$ direction to the image of the spherical harmonics. They are harmonic with respect to the Laplacian operator which allows for a relatively trivial transformation into the velocity space of a fluid through $\frac{1}{l}\nabla \times y_\ell^m$ where y_ℓ^m is a spherical harmonic basis function on the ℓth band.

The vorticity space is an infinite dimensional vector space, but the use of a finite number of spherical harmonics basis functions can provide a good approximation of the vorticity space for simulation purposes. In fact, the spherical harmonics are eigenfunctions on S^2 and by ordering them by eigenvalues in increasing order the later functions represent smaller scale vorticities, which provides the ability to chose the granularity of simulations. The ability to chose the granularity of the simulation allows a user to decide what sort of time vs performance trade off they want by choosing more or less basis functions or bands.

Because this method looks at an orthogonal family of basis functions the operators defining the change of the vector field representing a fluid over time, particularly the Lie bracket, can be calculated using second order polynomials of coefficients for spherical harmonics. This allows for all computations of changes of the vorticity field to be calculated in the space of spherical harmonics, and thus many computations necessary for the change of the vorticity over time can be reduced to the product of matrices. This is a significant improvement in contrast to most modern methods that require meshing of the fluid's domain and approximate velocity fields over a discretized point sets on a domain. Such methods often suffer from ailments that this method does not, such as the use of integration, which is both expensive and introduces numerical instability. With this method all that is required is pre-computation of several relationships between basis vector fields, particularly the Lie bracket which describes how the vorticity of each basis vector changes in reference to the velocity field defined by another basis vector.

First I give an overview of the necessary functions, properties, and some background for the derivation and then give the derivation itself.

2 Equations

2.1 Associated Legendre Polynomials

The associated Legendre Polynomials are a family of harmonic polynomials used in the definition of spherical harmonics. he associated Legendre polynomials are organized into bands, denoted by the subscript ℓ and within each band there are $2\ell + 1$ polynomials.
Rodrigues' Formula for associated Legendre polynomials:

$$P_\ell^m(x) = (-1)^m \ (1-x^2)^{m/2} \ \frac{\partial^m}{\partial x^m} \left(P_\ell(x)\right).$$

Part of the derivation that follows will require the ability to integrate triplets of associated Legendre polynomials. These triple integrals were first mentioned by Gaunt in [?], and an algorithm described in [?] calculates coefficients that correspond to the triple integrals in the following way:

$$\frac{2p+1}{2} \frac{(p-m-\mu)!}{p+m+\mu)!} \int_{-1}^{1} P_n^m P_\nu^\mu P_p^{m+\mu} dt = a(m,n,\nu,\mu,p).$$

$a(m,n,\nu,\mu,p)$ are the nonzero coefficients in the representation of the product of two associated Legendre polynomials as a linear combination of associated Legendre polynomials.

$$P_n^m P_\nu^\mu = \sum_{p=|n-\nu|}^{n+\nu} a(m,n,\nu,\mu,p) P_p^{m+\mu}.$$

I also use several recurrence relations from Edmond's Angular Momentum in Quantum Dynamics [?]:

$$\begin{aligned} P_{\ell+1}^m(x) - xP_\ell^m(x) - (\ell+m)(1-x^2)^{1/2} P_\ell^{m-1}(x) &= 0 \\ (1-x^2)^{1/2} P_\ell^{m+1}(x) - 2mxP_\ell^m(x) + (\ell+m)(\ell-m+1)(1-x^2)^{1/2} P_\ell^{m-1}(x) &= 0. \end{aligned}$$

2.2 Spherical Harmonics

Spherical harmonics is a harmonic basis for $\mathcal{S}^2 \to F$ where F is either $\mathbb{C}$ or $\mathbb{R}$. The complex spherical harmonics are defined as

$$y_\ell^m(\theta,\phi) = K_\ell^m P_\ell^m(\cos(\theta)) e^{im\phi}$$

$$K_\ell^m = \sqrt{\frac{(2\ell+1)}{4\pi} \frac{(\ell-|m|)!}{(\ell+|m|)!}}$$

where P_ℓ^m is an associated Legendre polynomial and K_ℓ^m is a normalization constant such that

$$\int_{S^2} y_\ell^m y_\nu^\mu = \delta_{\ell,\nu}\delta_{m,\mu}$$

with $\delta_{\alpha,\beta}$ being the Kronecker delta.
The real spherical harmonics are defined by:

$$y_\ell^m(\theta,\phi) = \begin{cases} \sqrt{2} K_\ell^m P_\ell^m(\cos(\theta))\cos(m\phi) & m > 0 \\ P_\ell(\cos(\theta)) & m = 0 \\ \sqrt{2} K_\ell^m P_\ell^{-m}(\cos(\theta))\sin(m\phi) & m < 0 \end{cases}.$$

To simplify the notation I introduce the $T_m : [0, 2\pi] \Rightarrow [-1,1]$ by

$$T_m(\phi) = \begin{cases} \sqrt{2}\cos(m\phi) & m > 0 \\ 1 & m = 0 \\ \sqrt{2}\sin(-m\phi) & m < 0 \end{cases}$$

which allows the real spherical harmonics to be defined by $y_\ell^m = K_\ell^m T_m P_\ell^{|m|}$ Sometimes spherical harmonics are parameterized with a $t \in [-1, 1]$ rather than to $\theta \in [0, \pi]$

$$y_\ell^m(t, \phi) = K_\ell^m T_{|m|}(\phi) P_\ell^{|m|}(t)$$

2.3 The Lie bracket

The Lie bracket measures the effects of one vector field upon another, It is notated as

$$[X, Y]_i = \sum_{j=1}^{n} \left(X_j \frac{\partial Y_i}{\partial x_j} - Y_j \frac{\partial X_i}{\partial x_j} \right).$$

2.4 Curl on $S^2 \to \mathbb{R}$

Since the vorticity vectors used only have $\hat{\boldsymbol{r}}$ components, all that is necessary is

$$\nabla \times A = \left(\tfrac{1}{\sin\theta} \tfrac{\partial A_r}{\partial \phi} \right) \hat{\boldsymbol{\theta}} + \left(-\tfrac{\partial A_r}{\partial \theta} \right) \hat{\boldsymbol{\phi}}.$$

2.5 Trigonometric Identities

These identities are necessary for solving the triple integral of sines and cosines.

$$\begin{aligned}
\cos(m\phi)\cos(\bar{m}\phi) &= \frac{\cos((m-\bar{m})\phi) + \cos((m+\bar{m})\phi)}{2} \\
\sin(m\phi)\sin(\bar{m}\phi) &= \frac{\cos((m-\bar{m})\phi) - \cos((m+\bar{m})\phi)}{2} \\
\sin(m\phi)\cos(\bar{m}\phi) &= \frac{\sin((m+\bar{m})\phi) + \sin((m-\bar{m})\phi)}{2} \\
\cos(m\phi)\sin(\bar{m}\phi) &= \frac{\sin((m+\bar{m})\phi) - \sin((m-\bar{m})\phi)}{2}
\end{aligned}$$

3 Derivations

3.1 Derivatives of spherical harmonics

If we differentiate an associated Legendre Polynomial from the Rodrigues formula we get:

$$\begin{aligned}
\frac{\partial P_\ell^m}{\partial x} &= (-1)^m\ (-2x)(\frac{m}{2}(1-x^2)^{m/2-1})\ \frac{\partial^m}{\partial x^m}(P_\ell) + (-1)^m\ (1-x^2)^{m/2}\ \frac{\partial^{m+1}}{\partial x^{m+1}}(P_\ell) \\
&= (\frac{-mx}{1-x^2})((1-x^2)^{m/2})\ (-1)^m\ \frac{\partial^m}{\partial x^m}(P_\ell) + (\frac{-1}{\sqrt{1-x^2}})(-1)^{m+1}\ (1-x^2)^{\frac{m+1}{2}}\ \frac{\partial^{m+1}}{\partial x^{m+1}}(P_\ell) \\
&= -\left(\frac{mx}{1-x^2}P_\ell^m + \frac{1}{\sqrt{1-x^2}}P_\ell^{m+1}\right).
\end{aligned}$$

The derivatives of the ϕ term in spherical harmonics is defined by

$$\begin{aligned}
\frac{\partial T_m}{\partial \phi} &= \begin{cases} -\sqrt{2}m\sin(m\phi) = mT_{-m}(\phi) & m > 0 \\ 0 & m = 0 \\ \sqrt{2}m\cos(m\phi) = mT_{-m}(\phi) & m < 0 \end{cases} \\
&= mT_{-m}(\phi).
\end{aligned}$$

The derivatives of real spherical harmonics are therefore as follows:

$$\begin{aligned}
\frac{\partial y_\ell^m}{\partial \phi} &= \frac{\partial}{\partial \phi}(K_\ell^m P_\ell^{|m|} T_m) \\
&= (K_\ell^m P_\ell^{|m|} \frac{\partial T_m}{\partial \phi}) \\
&= (K_\ell^m P_\ell^{|m|} m T_{-m}\phi) \\
&= m y_\ell^{-m} \\
\frac{\partial y_\ell^m}{\partial t} &= \frac{\partial}{\partial t}(K_\ell^m P_\ell^{|m|} T_m) \\
&= -K_\ell^m T_m \left(\frac{mt}{1-t^2}P_\ell^m + \frac{1}{\sqrt{1-t^2}}P_\ell^{m+1}\right).
\end{aligned}$$

3.2 Velocity Fields Corresponding to Spherical Harmonics Vorticities

Vorticity is a vector field normal S^2 where velocity vector field lies. Therefore it is natural to parameterize the vorticity as a vector field exclusively in the $\hat{r}$ direction. Since the dimensionality of the normal vector field to S^2 in $\mathbb{R}^3$ is 1, vorticity field can be mapped by radial spherical harmonics, which are the spherical harmonics applied to $\hat{\boldsymbol{r}}$

$$\hat{y}_\ell^m = y_\ell^m \hat{\boldsymbol{r}}$$

The velocity basis field Φ_k can be defined as

$$\Phi_k = \nabla \times (\Delta^{-1}\phi_k) = \nabla \times (\frac{1}{\lambda_k}\phi_k) = \frac{1}{\lambda_k}(\nabla \times \phi_k).$$

Therefore the velocity basis field can be explicitly declared as

$$\begin{aligned}
\Phi_\ell^m = \frac{1}{\lambda_\ell^m}\nabla \times y_\ell^m &= \frac{1}{\ell}\left(\left(\frac{1}{\sin\theta}\frac{\partial y_\ell^m}{\partial \phi}\right)\hat{\boldsymbol{\theta}} + \left(-\frac{\partial y_\ell^m}{\partial\theta}\right)\hat{\boldsymbol{\phi}}\right) \\
&= \frac{1}{\ell}\left(\left(\frac{m}{\sin\theta}y_\ell^{-m}\right)\hat{\boldsymbol{\theta}} + \left(K_\ell^m T_m\left(m\frac{t}{1-t^2}P_\ell^m + \frac{1}{\sqrt{1-t^2}}P_\ell^{m+1}\right)(\frac{\partial t}{\partial \theta})\right)\hat{\boldsymbol{\phi}}\right) \\
&= \frac{1}{\ell}\left(\left(\frac{m}{\sin\theta}y_\ell^{-m}\right)\hat{\boldsymbol{\theta}} + \left(K_\ell^m T_m\left(m\frac{\cos\theta}{\sin^2\theta}P_\ell^m + \frac{1}{|\sin\theta|}P_\ell^{m+1}\right)(-\sin\theta)\right)\hat{\boldsymbol{\phi}}\right) \\
&= \frac{1}{\ell}\left(\left(\frac{m}{\sin\theta}y_\ell^{-m}\right)\hat{\boldsymbol{\theta}} - \left((m\cot\theta)y_\ell^m + (sign(\sin\theta))K_\ell^m P_\ell^{m+1}T_m\right)\hat{\boldsymbol{\phi}}\right).
\end{aligned}$$

3.3 The Lie Bracket on S^2

The Lie bracket $[A, B]$ represents the effect of B upon A. If we let A be a velocity field on S^2, it must be tangent to S^2 and have degenerate entries in the $\hat{\boldsymbol{r}}$ direciton. Similarly if we let B be normal to a velocity field on S^2 it must be normal to S^2 and have degenerate entries in the $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\phi}}$ directions. Therefore when we expand the Lie bracket we get the following for the α direction of the Lie bracket:

$$\begin{aligned}
[A, B]_\alpha &= \sum_{\beta\in\{\hat{\theta},\hat{\phi}\}}\left(A_\beta\frac{\partial B_\alpha}{\partial\beta} - B_\beta\frac{\partial A_\alpha}{\partial\beta}\right) \\
&= A_\theta\frac{\partial B_\alpha}{\partial\theta} + A_\phi\frac{\partial B_\alpha}{\partial\phi}.
\end{aligned}$$

Half of the terms above disappear because B is degenerate in $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\phi}}$. This also means $\beta \in \{\hat{\theta}, \hat{\phi}\}$, $\frac{\partial B}{\partial \beta} = 0$ as well.

$$\begin{aligned}
[A, B]_{\hat{\theta}} &= A_\theta\frac{\partial B_\theta}{\partial\theta} + A_\phi\frac{\partial B_\theta}{\partial\phi} \\
&= 0
\end{aligned}$$

$$\begin{aligned}
[A, B]_{\hat{\phi}} &= A_\theta\frac{\partial B_\phi}{\partial\theta} + A_\phi\frac{\partial B_\phi}{\partial\phi} \\
&= 0
\end{aligned}$$

$$[A, B]_{\hat{r}} = A_\theta\frac{\partial B_r}{\partial\theta} + A_\phi\frac{\partial B_r}{\partial\phi}.$$

In the future a lack of subscript on a Lie bracket assumes the $\hat{\boldsymbol{r}}$ direction.

3.4 Calculation of Coefficient Matrix Entries

The entries of the coefficient matrices $C_{\phi_\alpha}[\phi_i, \phi_j]$ in this formulation are defined by the change in vorticity field ϕ_j on a particle moving along ϕ_i and ϕ_j by means of the Lie bracket $[\frac{1}{\lambda_i}\nabla \times \phi_i, \phi_j]$. The coefficient matrix entry $C_{\phi_\alpha}[\phi_i, \phi_j] = a_i$ in the unique expansion of $[\frac{1}{\lambda_i}\nabla \times \phi_i, \phi_j]$ given by

$$[\frac{1}{\lambda_i}\nabla \times \phi_i, \phi_j] = \sum_{\alpha=0}^{\infty}\phi_\alpha.$$

$$
\begin{aligned}
[\nabla \times \hat{y}_\ell^m, \hat{y}_{\ell'}^{m'}] &= \frac{1}{\sin\theta}\frac{\partial y_\ell^m}{\partial \phi}\frac{\partial y_{\ell'}^{m'}}{\partial \theta} - \frac{\partial y_\ell^m}{\partial \theta}\frac{\partial y_{\ell'}^{m'}}{\partial \phi} \\
&= \frac{1}{\sin\theta}\frac{\partial y_{\ell'}^{m'}}{\partial \phi}\frac{\partial y_\ell^m}{\partial t}\frac{\partial t}{\partial \theta} - \frac{\partial y_\ell^m}{\partial \phi}\frac{\partial y_{\ell'}^{m'}}{\partial t}\frac{\partial t}{\partial \theta} \\
&= \left(\frac{1}{\sin\theta}\frac{\partial y_{\ell'}^{m'}}{\partial \phi}\frac{\partial y_\ell^m}{\partial t} - \frac{\partial y_\ell^m}{\partial \phi}\frac{\partial y_{\ell'}^{m'}}{\partial t}\right)\frac{\partial t}{\partial \theta} \\
&= \frac{1}{\sin\theta}(m' y_{\ell'}^{-m'})\left(-K_\ell^m T_m\left(\frac{mt}{1-t^2}P_\ell^m + \frac{1}{\sqrt{1-t^2}}P_\ell^{m+1}\right)\right)(-\sin\theta) \\
&\quad -(m y_\ell^{-m})\left(-K_{\ell'}^{m'} T_m\left(\frac{m't}{1-t^2}P_{\ell'}^{m'} + \frac{1}{\sqrt{1-t^2}}P_{\ell'}^{m'+1}\right)\right)(-\sin\theta) \\
&= (m' y_{\ell'}^{-m'})\left(K_\ell^m T_m\left(\frac{mt}{1-t^2}P_\ell^m + \frac{1}{\sqrt{1-t^2}}P_\ell^{m+1}\right)\right) \\
&\quad -(m y_\ell^{-m})\left(K_{\ell'}^{m'} T'_m\left(\frac{m't}{1-t^2}P_{\ell'}^{m'} + \frac{1}{\sqrt{1-t^2}}P_{\ell'}^{m'+1}\right)\sqrt{1-t^2}\right) \\
&= (m' y_{\ell'}^{-m'})\left(K_\ell^m T_m\left(\frac{mt}{1-t^2}P_\ell^m + \frac{1}{\sqrt{1-t^2}}P_\ell^{m+1}\right)\right) \\
&\quad -(m y_\ell^{-m})\left(K_{\ell'}^{m'} T_m\left(\frac{m't}{\sqrt{1-t^2}}P_{\ell'}^{m'} + P_{\ell'}^{m'+1}\right)\right)
\end{aligned}
$$

Since spherical harmonics are orthonormal we can integrate the above equation with a spherical harmonics basis function's dual basis, $\bar{y}_\nu^\mu$ to calculate $C_{y_\nu^\mu}[y_\ell^m, y_{\ell'}^{m'}] = \alpha_\nu^\mu$ from the summation

$$
[\frac{1}{l}\nabla \times \hat{y}_\ell^m, \hat{y}_{\ell'}^{m'}] = \sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu} a_\nu^\mu y_\nu^\mu
$$

Since we are really dealing with real spherical harmonics the conjugate becomes unneccessary. The calculation is as follows:

$$
\begin{aligned}
a_\nu^\mu &= \int_0^{2\pi}\int_0^{\pi} y_\nu^\mu [\frac{1}{l}\nabla \times \hat{y}_\ell^m, \hat{y}_{\ell'}^{m'}] d\theta d\phi \\
&= \int_0^{2\pi}\int_0^{\pi} y_\nu^\mu \left(\frac{1}{\sin\theta}\frac{\partial y_{\ell'}^{m'}}{\partial \phi}\frac{\partial y_\ell^m}{\partial t} - \frac{\partial y_\ell^m}{\partial \phi}\frac{\partial y_{\ell'}^{m'}}{\partial t}\right)\frac{\partial t}{\partial \theta} d\theta d\phi \\
&= \frac{1}{\ell}\int_0^{2\pi}\int_{-1}^{1}\left(\begin{array}{c} (m' y_{\ell'}^{-m'})\left(K_\ell^m T_m\left(\frac{mt}{1-t^2}P_\ell^m + \frac{1}{\sqrt{1-t^2}}P_\ell^{m+1}\right)\right) \\ -(m y_\ell^{-m})\left(K_{\ell'}^{m'} T_m\left(\frac{m't}{\sqrt{1-t^2}}P_{\ell'}^{m'} + P_{\ell'}^{m'+1}\right)\right)\end{array}\right) y_\nu^\mu dt d\phi \\
&= \frac{1}{\ell}\int_0^{2\pi}\int_{-1}^{1}(m' y_{\ell'}^{-m'}) K_\ell^m T_m\left(\frac{mt}{1-t^2}P_\ell^m + \frac{1}{\sqrt{1-t^2}}P_\ell^{m+1}\right) y_\nu^\mu dt d\phi \\
&\quad -\frac{1}{\ell}\int_0^{2\pi}\int_{-1}^{1}\left((m y_\ell^{-m}) K_{\ell'}^{m'} T_m\left(\frac{m't}{\sqrt{1-t^2}}P_{\ell'}^{m'} + P_{\ell'}^{m'+1}\right)\right) y_\nu^\mu dt d\phi \\
&= \frac{1}{\ell}K_\ell^m K_{\ell'}^{m'} K_\nu^\mu \int_0^{2\pi} T_m T_{-m'} T_\mu d\phi \int_{-1}^{1} m'\left(\frac{mt}{1-t^2}P_\ell^m + \frac{1}{\sqrt{1-t^2}}P_\ell^{m+1}\right) P_{\ell'}^{m'} P_\nu^\mu dt \\
&\quad -K_\ell^m K_{\ell'}^{m'} K_\nu^\mu \int_0^{2\pi} T_{-m} T_{m'} T_\mu d\phi \int_{-1}^{1} m\left(\frac{m't}{\sqrt{1-t^2}}P_{\ell'}^{m'} + P_{\ell'}^{m'+1}\right) P_\ell^m P_\nu^\mu dt
\end{aligned}
$$

The coefficient for the projection of the Lie bracket onto spherical harmonics can be determined by the above integral, but it's not quite in a form that's easily computed. The individual components, howerver, are easily decomposed into several easily computed components. The first and most obvious part is the independence of the ϕ and t dependent portions of the above integral.
The derivations of the t components require a bit more refining through a combination of applications of the Edmonds recurrence relations to reformulate the integrand as the linear combination of associated Legendre polynomial triple products. Once reformulated as a linear combination of triple products the integrals can easily be computed through an algorithm to compute Gaunt coefficients. The Edmonds recurrence relations give, with the exception of a few situations, ways to expand the t terms that were not immediately linear combinations of associated Legendre Polynomials into such functions.

$$\frac{t}{\sqrt{1-t^2}}P_\ell^m = \frac{1}{2m}\left(P_\ell^{m+1} + (\ell+m)(\ell-m+1)P_\ell^{m-1}\right)$$

$$\begin{aligned}
\frac{1}{\sqrt{1-t^2}}P_\ell^m &= \frac{t}{\sqrt{1-t^2}}P_{(\ell-1)}^m + ((\ell-1)+m)P_{(\ell-1)}^{m-1} \\
&= \frac{1}{2m}\left(P_{(\ell-1)}^{m+1} + ((\ell-1)+m)((\ell-1)-m+1)P_{(\ell-1)}^{m-1}\right) + ((\ell-1)+m)P_{(\ell-1)}^{m-1} \\
&= \frac{1}{2m}\left(P_{\ell-1}^{m+1} + (\ell+m-1)(\ell-m)P_{\ell-1}^{m-1}\right) + (\ell+m-1)P_{\ell-1}^{m-1} \\
&= \frac{1}{2m}\left(P_{\ell-1}^{m+1} + (\ell+m-1)(\ell+m)P_{\ell-1}^{m-1}\right)
\end{aligned}$$

The above relations cease to work if $m=0$ but one must note that spherical harmonics only use associated Legendre Polynomials where $m \geq 0$ and in the cases where m or m' could be 0 (such as $\frac{mt}{\sqrt{1-t^2}}P_\ell^m$) the recurrence relations are unnecessary as $m=0$ and so the integral of that component is 0. Once the terms are expanded we have a sum of triple products of associated Legendre Polynomials ($P_n^m P_{n'}^{m'} P_\nu^\mu$) which can then be calculated using Gaunt coefficients.
Since the product of trigonometric functions can be reduced to the sum of two trigonometric functions and with $m \in \mathbb{N}$ they form an orthogonal basis on $[0, 2\pi]$, First, here's the different combinations of $T_m \frac{\partial T_{\bar{m}}}{\partial \phi}$ that can happen:

$T_m \frac{\partial T_{\bar{m}}}{\partial \phi}$	
$-\bar{m}\cos(m\phi)\sin(\bar{m}\phi)$	$m>0, \bar{m}>0$
$\bar{m}\cos(m\phi)\cos(\bar{m}\phi)$	$m>0, \bar{m}<0$
$-\bar{m}\sin(m\phi)\sin(\bar{m}\phi)$	$m<0, \bar{m}>0$
$\bar{m}\sin(m\phi)\cos(\bar{m}\phi)$	$m<0, \bar{m}<0$

Since the integral has been reduced to the integral of two trig functions and we know that the family that we're looking at has integral $m, \bar{m}, \mu$ coefficients, they are an orthogonal family of polynomials and the integral of two such trig functions is π. Therefore the following is the solution to all of the cases

$$m=0 \rightarrow T_m \frac{\partial T_{\bar{m}}}{\partial \phi} := \operatorname{sgn}(\bar{m})\bar{m}\pi\delta_{\bar{m},\mu} \quad \bar{m}=0 \rightarrow T_m \frac{\partial T_{\bar{m}}}{\partial \phi} := 0$$

$T_m \frac{\partial T_{\bar{m}}}{\partial \phi}$	
$-\frac{\pi\bar{m}}{2}(-\delta_{m+\bar{m},-\mu} - \operatorname{sgn}(m-\bar{m})\delta_{\lvert m-\bar{m}\rvert,-\mu})$	$m>0, \bar{m}>0$
$-\frac{\pi\bar{m}}{2}(\delta_{\lvert m-\bar{m}\rvert,\mu} + \delta_{\lvert m+\bar{m}\rvert,\mu})$	$m>0, \bar{m}<0$
$-\frac{\pi\bar{m}}{2}(\delta_{\lvert m-\bar{m}\rvert,\mu} - \delta_{\lvert m+\bar{m}\rvert,\mu})$	$m<0, \bar{m}>0$
$-\frac{\pi\bar{m}}{2}(\delta_{m+\bar{m},\mu} + \operatorname{sgn}(m-\bar{m})\delta_{\lvert m-\bar{m}\rvert,-\mu})$	$m<0, \bar{m}<0$