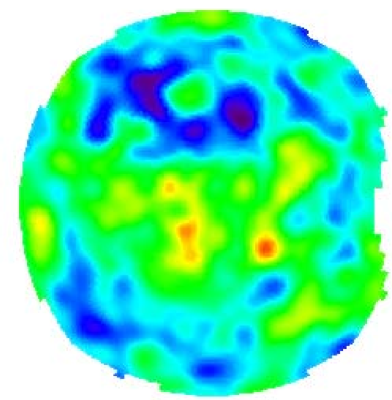
Orthogonal and Symmetric Haar Wavelets on the Sphere

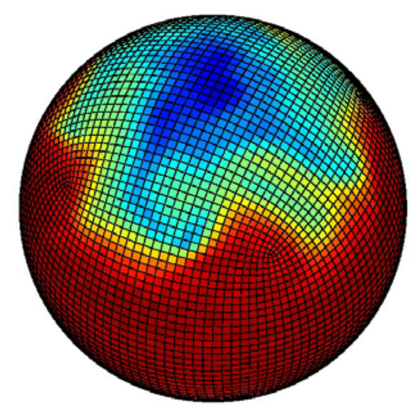
UNIVERSITY of TORONTO

dgp
dynamic graphics project

Motivation

- Spherically parametrized signals can be found in many fields:
  - Computer graphics
  - Physics
  - Astronomy
  - Climate modeling
  - Medical imaging
  - ...





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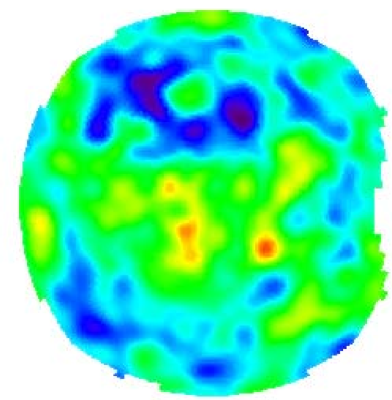
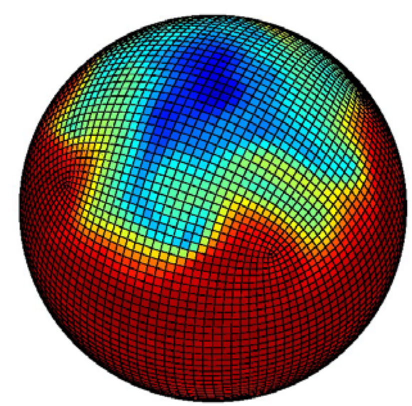
- An efficient representation of spherical signals is of importance for many applications





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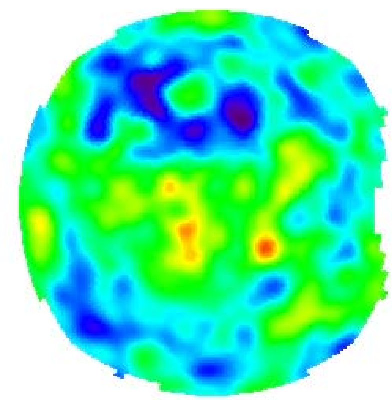

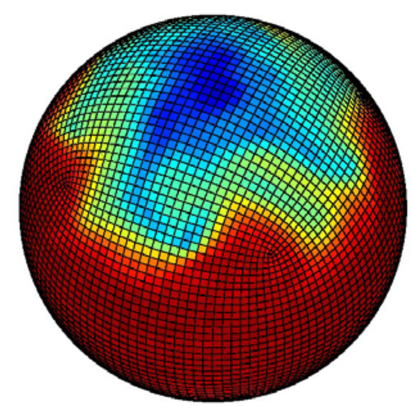

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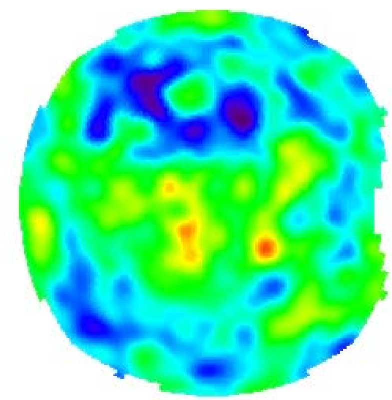
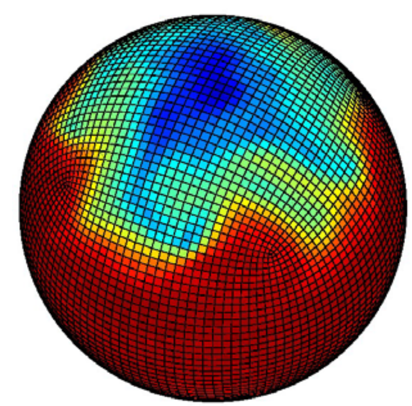
  - process signals efficiently in the basis representation





Requirements

1. Localization in space and frequency
   – Necessary condition for the efficient representation of arbitrary (all-frequency) signals

Requirements

1. Localization in space and frequency
2. Very compact support
   – Minimal costs for basis projection and reconstruction
   – Computations in the basis representation are efficient

Requirements

1. Localization in space and frequency
2. Very compact support
3. Orthogonality
   – ℓ_2 optimal approximation can be found efficiently
   – Computations are in many cases more efficient

Requirements

1. Localization in space and frequency
2. Very compact support
3. Orthogonality
4. Orientation-free representation
   - Avoids artifacts when a signal is approximated in the basis representation

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Representations for Spherical Signals

- Spherical Harmonics [MacRobert 1948]
  - Physics and chemistry [Edmonds 1957]
  - Geoscience [Clarke 2004]
  - Medical imaging [Katsuyuki 2001]
  - Computer graphics [Cabral 1987, Westin 1992, Ramamoorthi 2002, Kautz 2002, Sloan 2002]

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- Spherical Radial Basis Functions
  - Astronomy and geoscience [Fisher 1993; Freeden 1998]
  - Computer graphics [Green 2006; Tsai 2006]

Wavelets for Spherical Signals

- Wavelets defined over Euclidean domains
  [Lalonde 1997; Ng 2003; Zhou 2005; Wang 2006; Sun 2006]
- Wavelet bases defined over subdivision surfaces
  [Lounsbery 1997]
- Haar wavelets for general measure spaces L_p [Girardi 1997]
- Discrete spherical wavelets [Schröder 1995]
- Nearly orthogonal spherical Haar wavelets [Nielson 1997; Bonneau 1999; Rosça 2004]
- Pseudo wavelets defined over the sphere [Ma 2006]

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Contributions

- Development of SOHO wavelet basis
  - Constructive proof
  - Proof that SOHO wavelets form an unconditional basis of $L_2(\mathbb{S}^2, d\omega)$

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- Development of rotation matrices for spherical Haar wavelet bases
  - Analytic computation of matrix elements
- Experimental evaluation
  - Important for practicality of SOHO wavelets

Spherical Wavelets: Design Space

- Subdivision scheme

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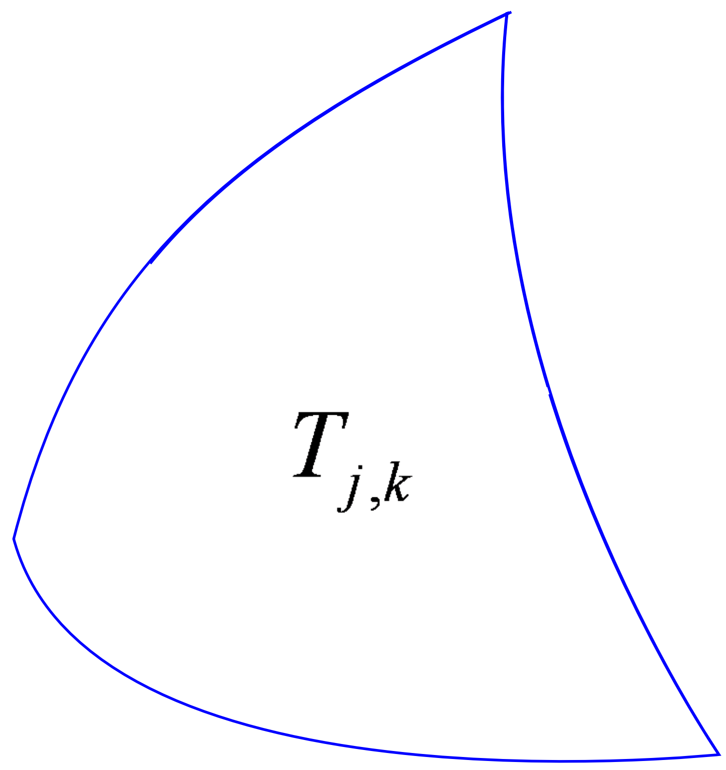
- Refinement filter coefficients

$$\begin{pmatrix} \lambda^0_{j+1} \\ \lambda^1_{j+1} \\ \lambda^2_{j+1} \\ \lambda^3_{j+1} \end{pmatrix} = \underbrace{\begin{pmatrix} h_0 & g^0_0 & g^1_0 & g^2_0 \\ h_1 & g^0_1 & g^1_1 & g^2_1 \\ h_2 & g^0_2 & g^1_2 & g^2_2 \\ h_3 & g^0_3 & g^1_3 & g^2_3 \end{pmatrix}}_{S_{j,k}} \begin{pmatrix} \lambda_j \\ \gamma^0_j \\ \gamma^1_j \\ \gamma^2_j \end{pmatrix}$$

Subdivision Scheme

- Previous work employed geodesic bisector subdivision



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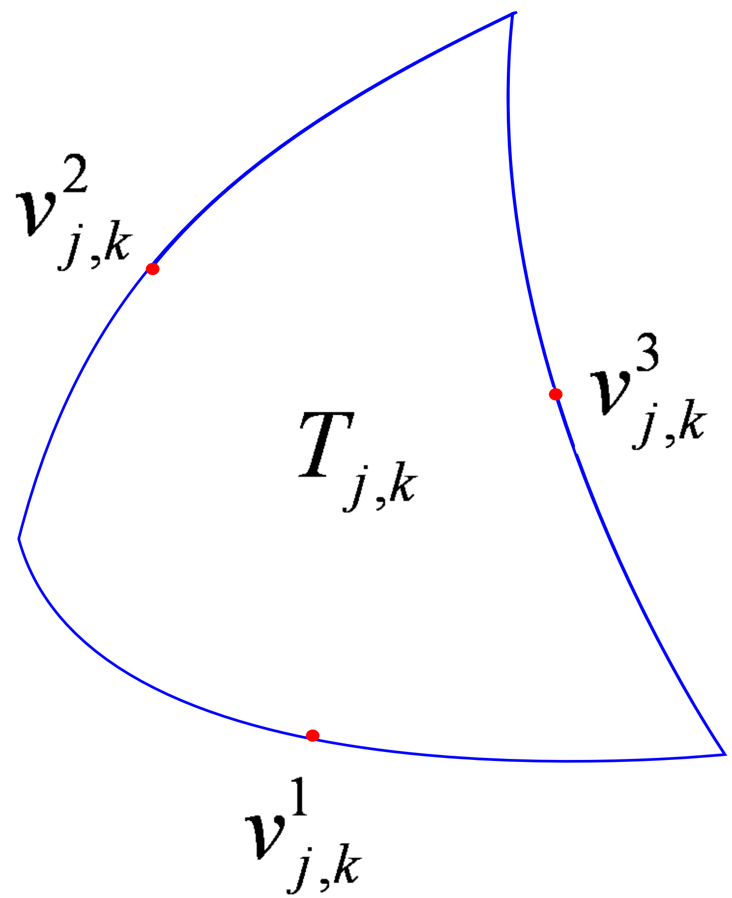

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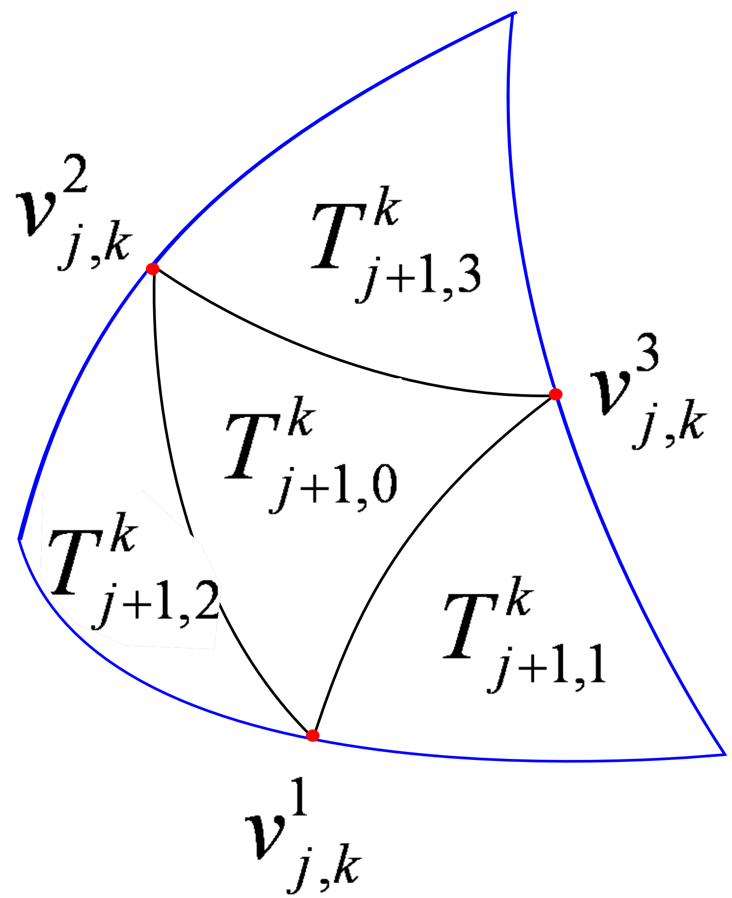

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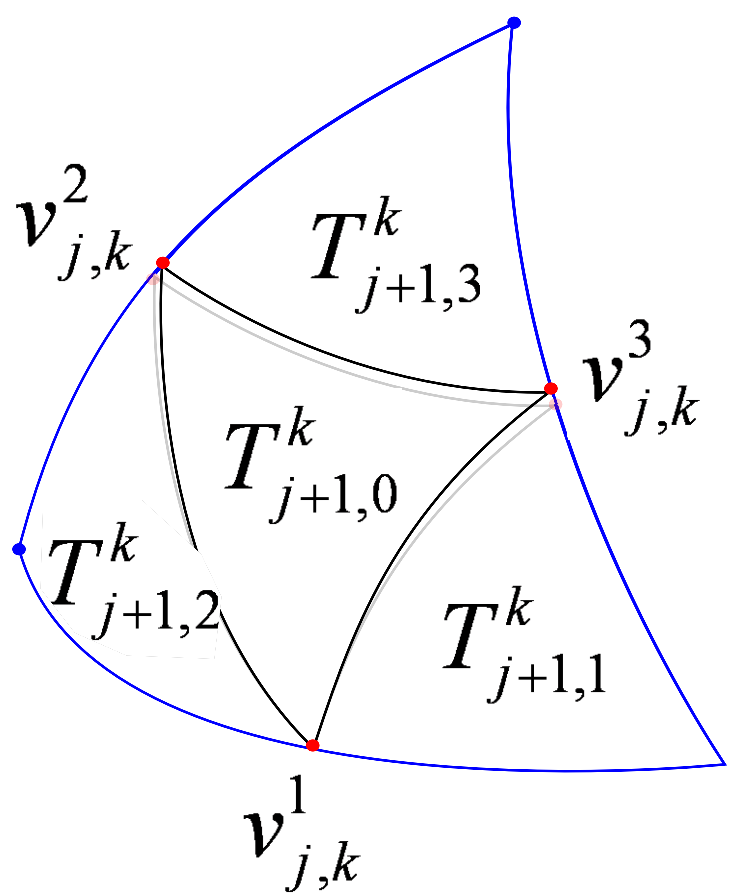
  - Orthogonal and symmetric spherical Haar wavelet basis *probably* does not exist
- 4-fold subdivision required for partition
  - BUT not that geodesic bisector is employed

Subdivision Scheme

- Subdivision scheme can be defined so that the area of the three outer child triangles be equal



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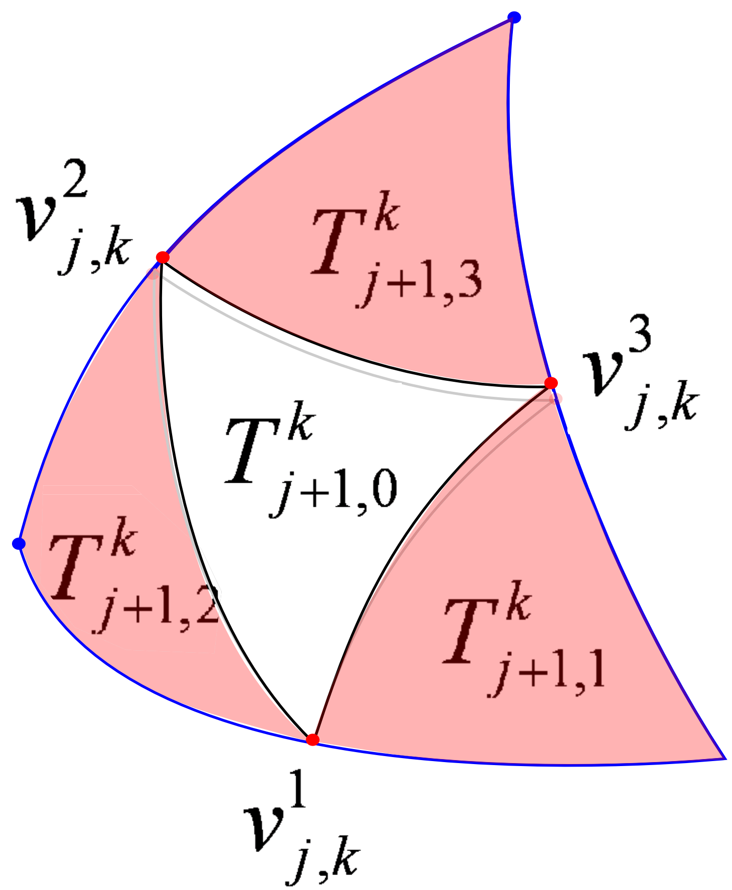
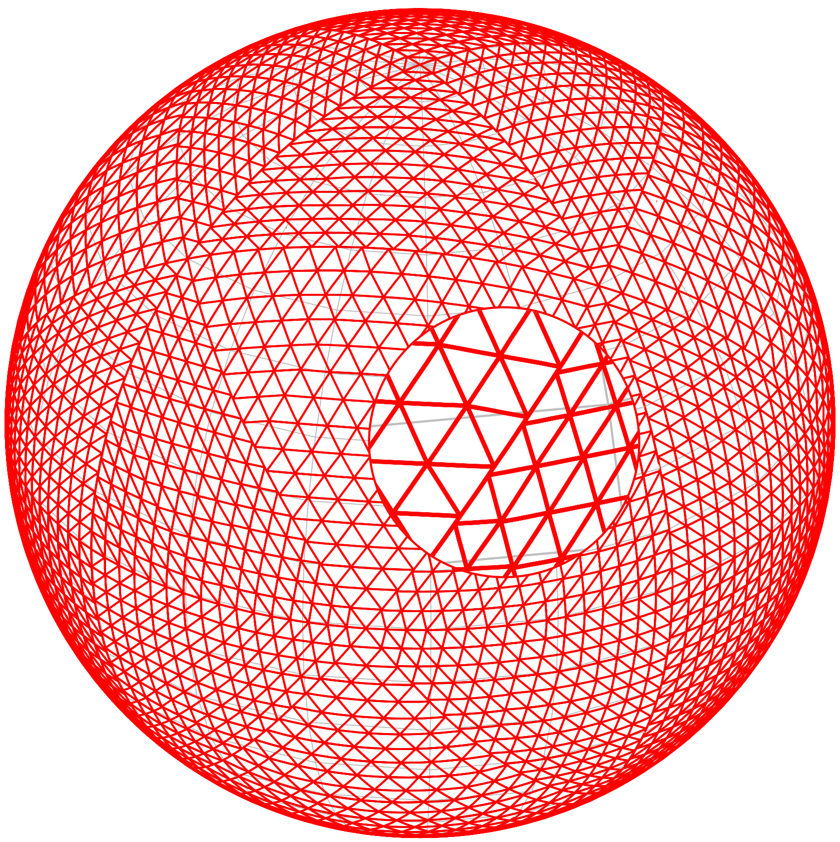
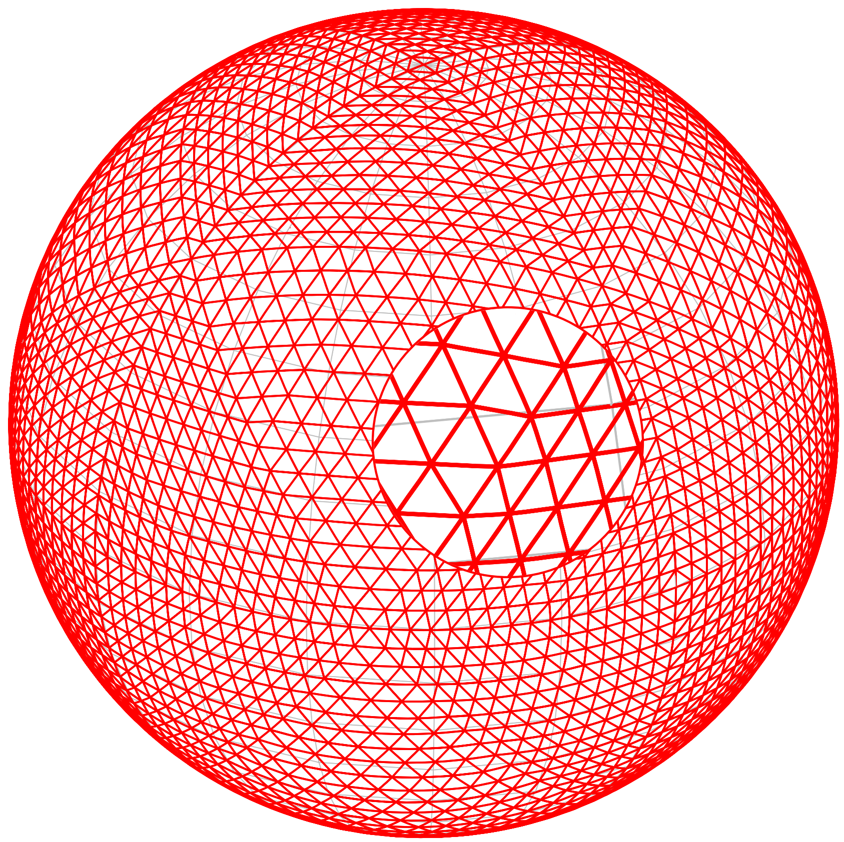


Subdivision Scheme



Our subdivision



Geodesic Bisector

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Local Reconstruction Relationship

- Parametric synthesis matrix

$$\hat{S}_{j,k} = \begin{pmatrix} \frac{\sqrt{\alpha_0}}{\sqrt{\alpha_p}} & -c\frac{\sqrt{\alpha_1}}{\sqrt{\alpha_0}} & -c\frac{\sqrt{\alpha_1}}{\sqrt{\alpha_0}} & -c\frac{\sqrt{\alpha_1}}{\sqrt{\alpha_0}} \\ \frac{\sqrt{\alpha_1}}{\sqrt{\alpha_p}} & b & a & a \\ \frac{\sqrt{\alpha_1}}{\sqrt{\alpha_p}} & a & b & a \\ \frac{\sqrt{\alpha_1}}{\sqrt{\alpha_p}} & a & a & b \end{pmatrix}$$

SOHO Wavelet Basis

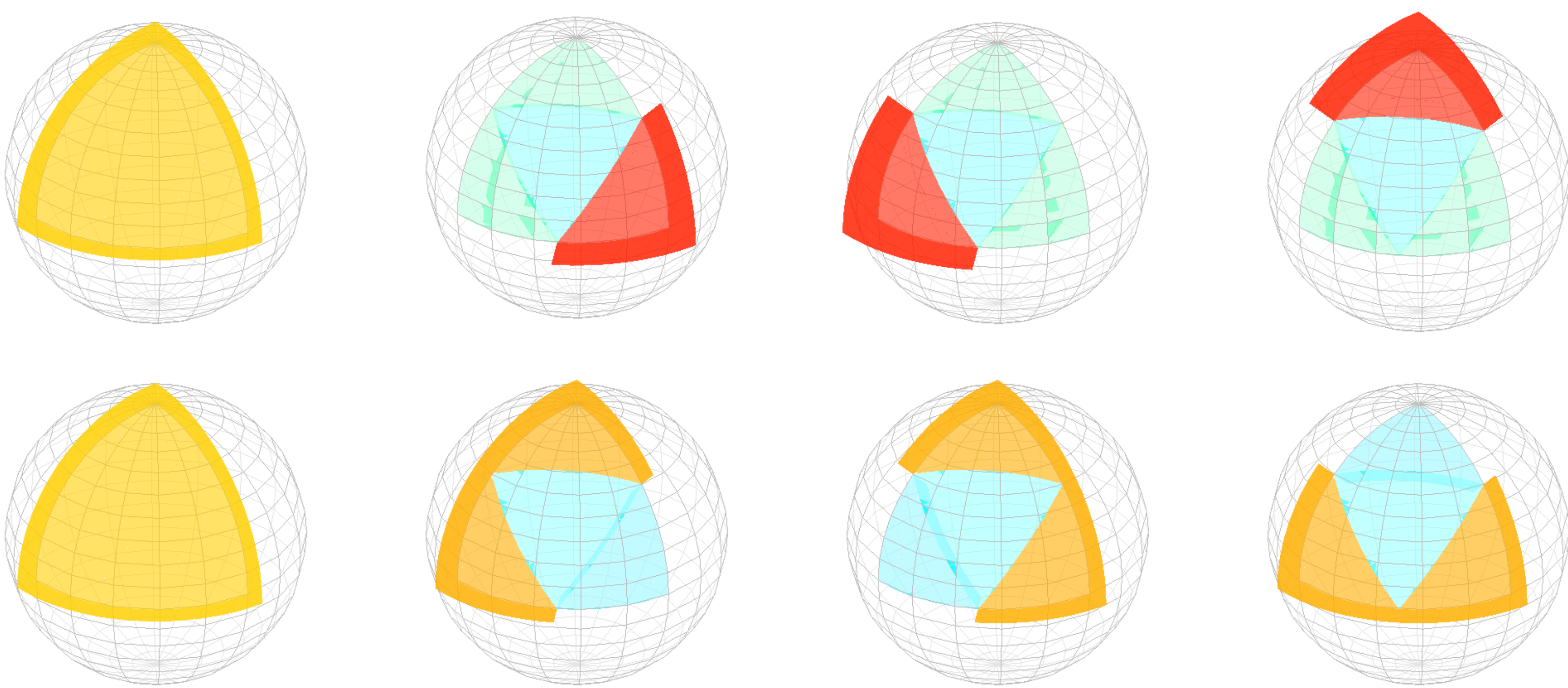
- Synthesis matrix

$$S_{j,k} = \begin{pmatrix} \frac{\sqrt{\alpha_0}}{\sqrt{\alpha_p}} & \frac{\sqrt{\alpha_1}}{\sqrt{\alpha_0}} & \frac{\sqrt{\alpha_1}}{\sqrt{\alpha_0}} & \frac{\sqrt{\alpha_1}}{\sqrt{\alpha_0}} \\ \frac{\sqrt{\alpha_1}}{\sqrt{\alpha_p}} & \frac{-2a+1}{\sqrt{\alpha_1}} & a & a \\ \frac{\sqrt{\alpha_1}}{\sqrt{\alpha_p}} & a & \frac{-2a+1}{\sqrt{\alpha_1}} & a \\ \frac{\sqrt{\alpha_1}}{\sqrt{\alpha_p}} & a & a & \frac{-2a+1}{\sqrt{\alpha_1}} \end{pmatrix}$$

$$a = \alpha_0 \pm \frac{\sqrt{\alpha_0^2 + 3\alpha_0\alpha_1}}{3\alpha_0}$$

SOHO Wavelet Basis

- Basis functions

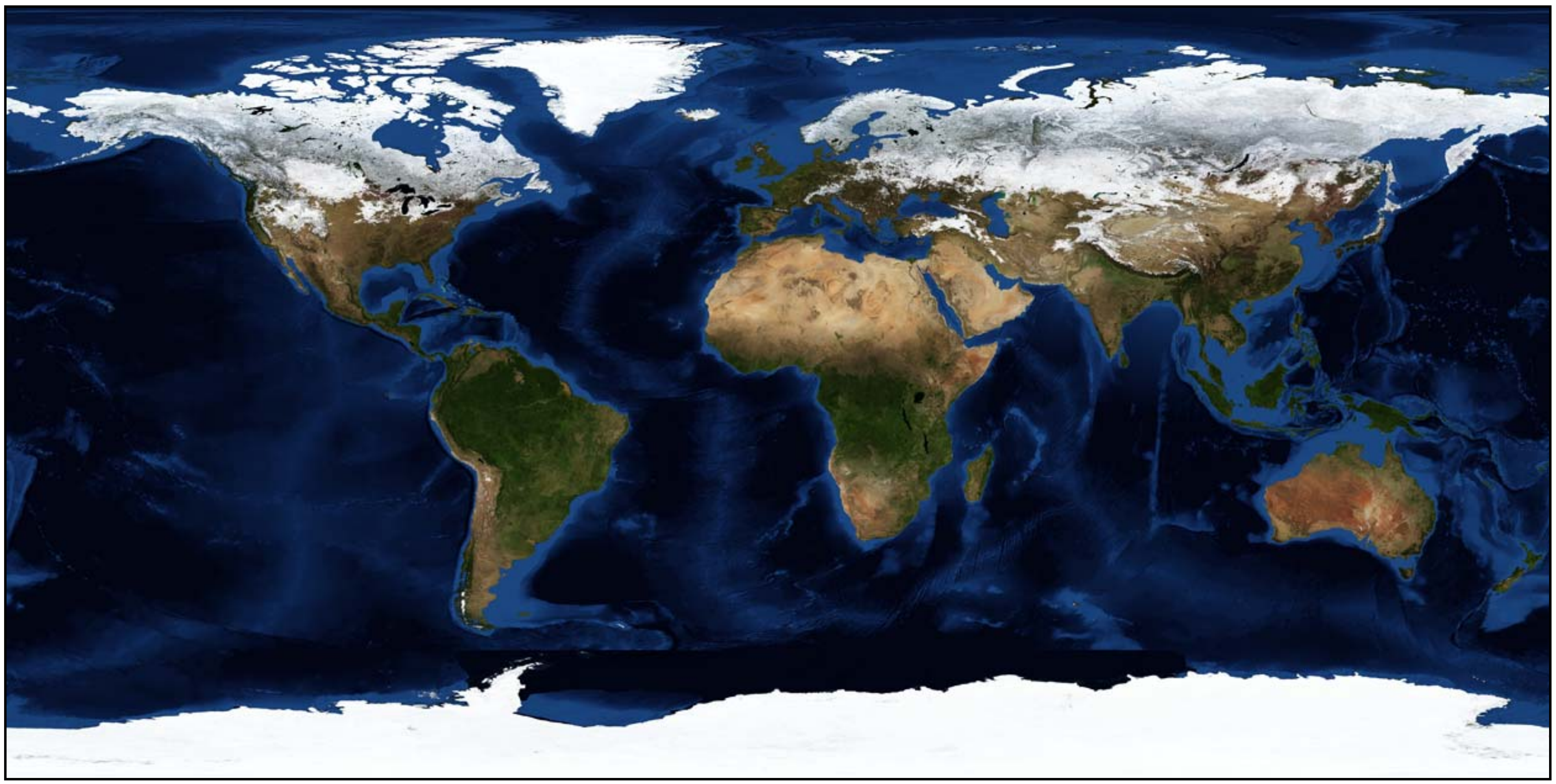


Experiments

- Comparison of SOHO wavelet basis with six previously proposed spherical Haar wavelet bases
  - Bio-Haar wavelets [Schröder 1995]
  - Pseudo Haar wavelets [Ma 2006]
  - Four nearly orthogonal spherical Haar wavelet bases [Nielson 1997; Bonneau 1999]

Experiments

- Three test signals



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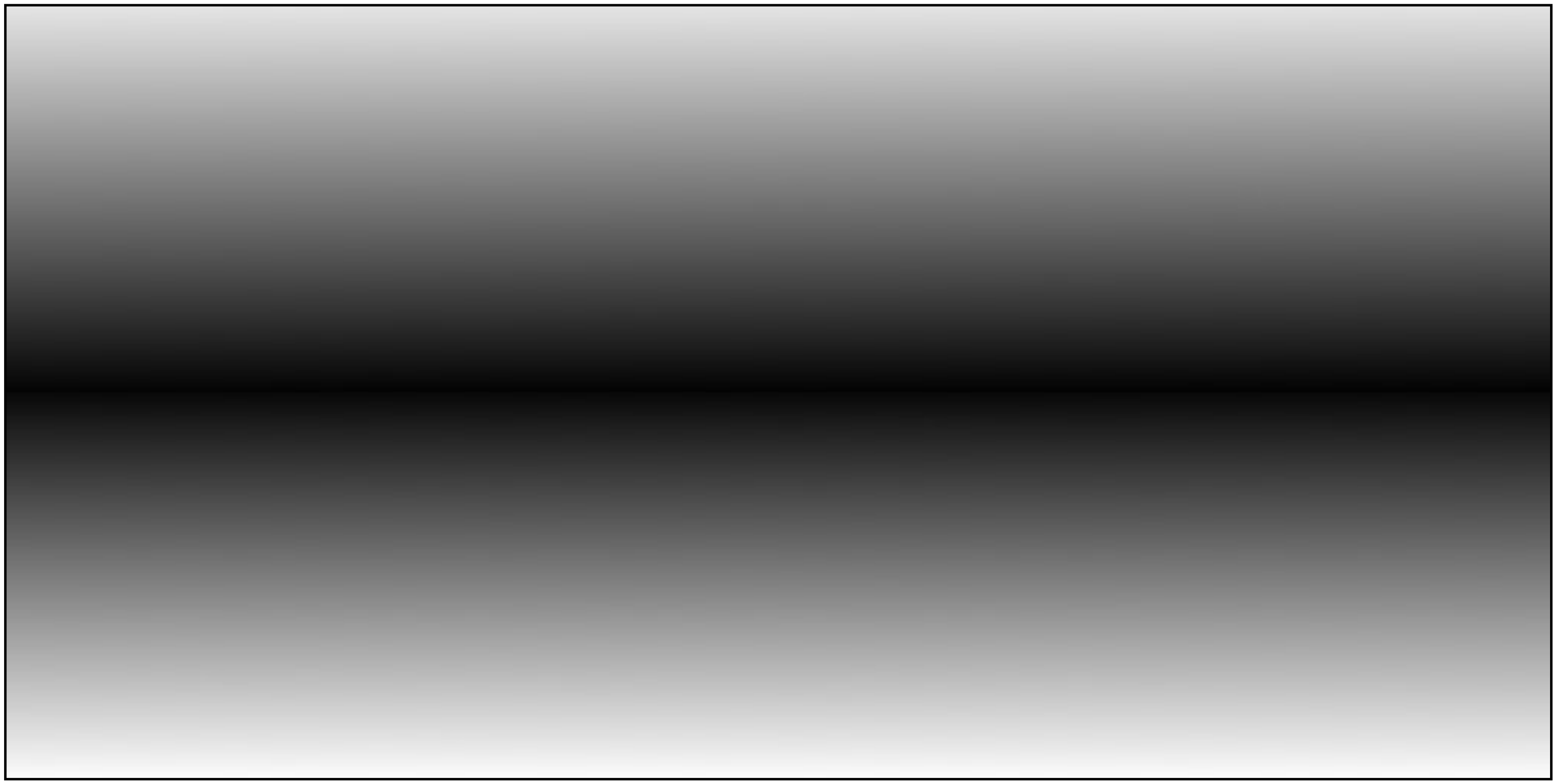

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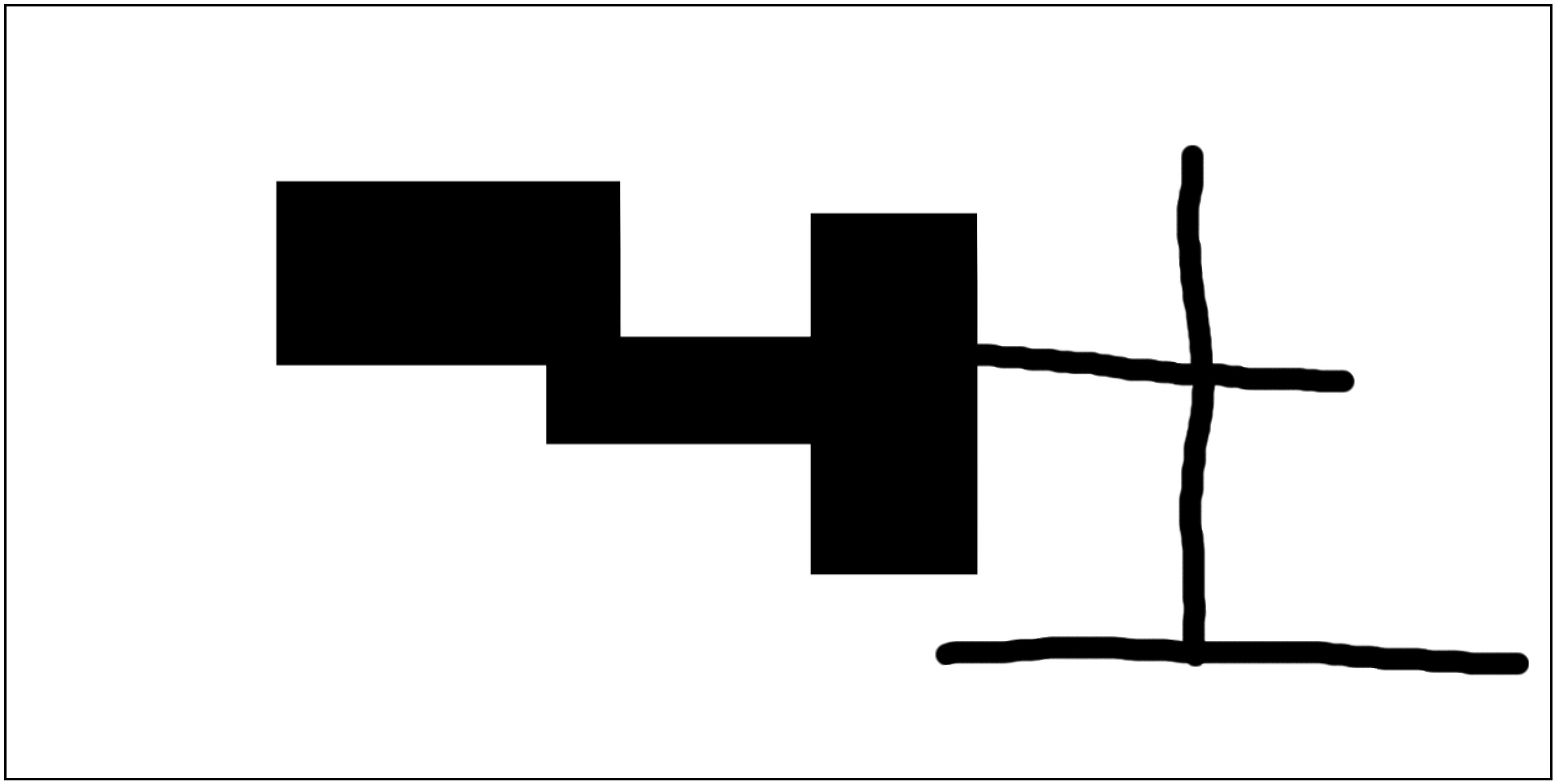


Experiments

- Nonlinear approximation: Reconstruction of the test signals with a subset of all basis function coefficients
- Increasing number of nonzero basis function coefficients
- ℓ_2 optimal approximation strategies
- Octahedron as base polyhedron
- Signals defined over partitions on level eight of the partition trees: $131{,}072$ basis function coefficients

Experiments

- Number of basis function coefficients: 64

Experiments

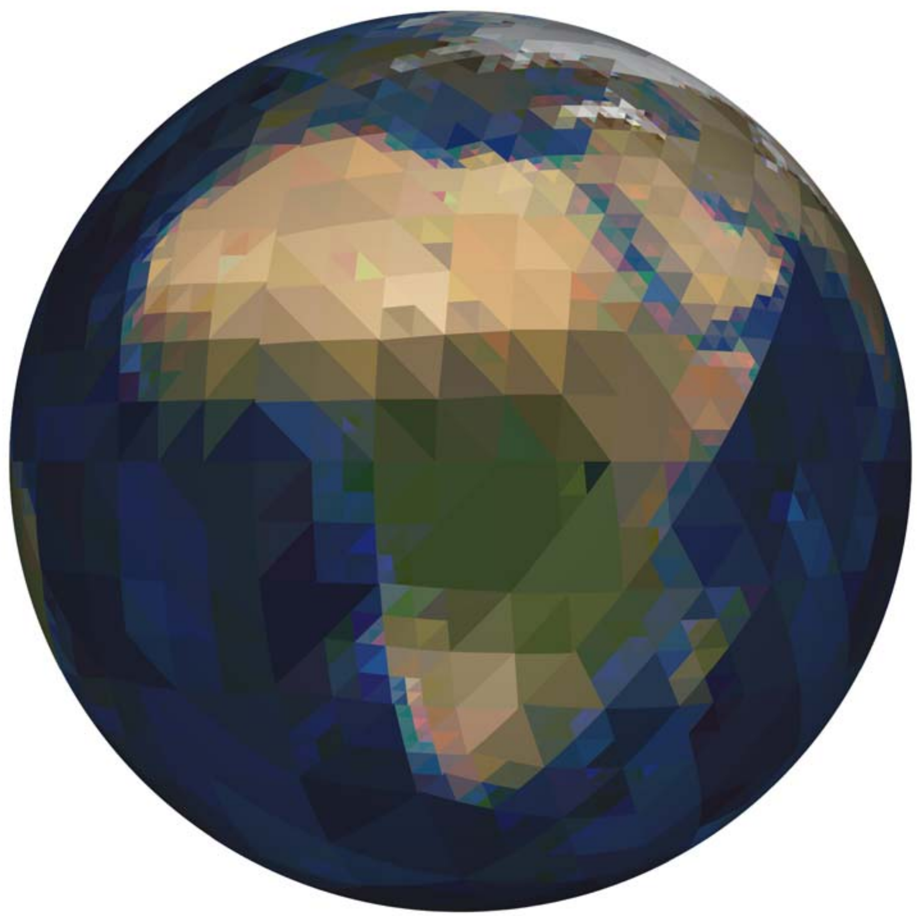
- Number of basis function coefficients: 256

Experiments

- Number of basis function coefficients: 512

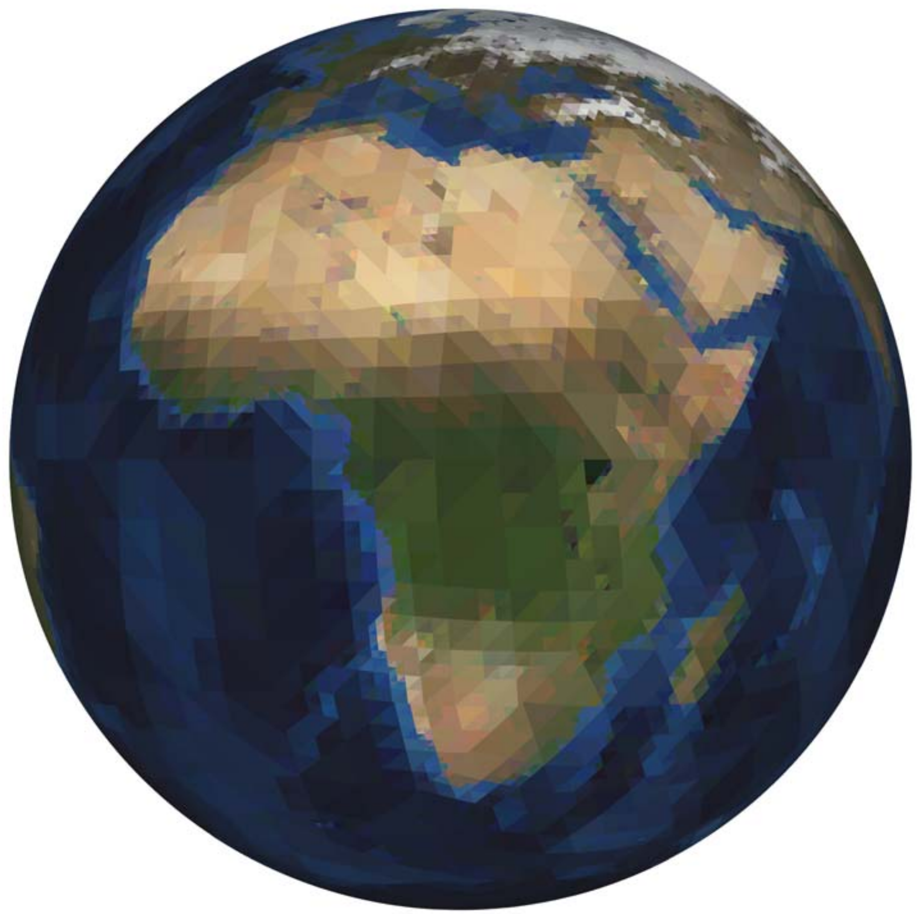
Experiments

- Number of basis function coefficients: 2056



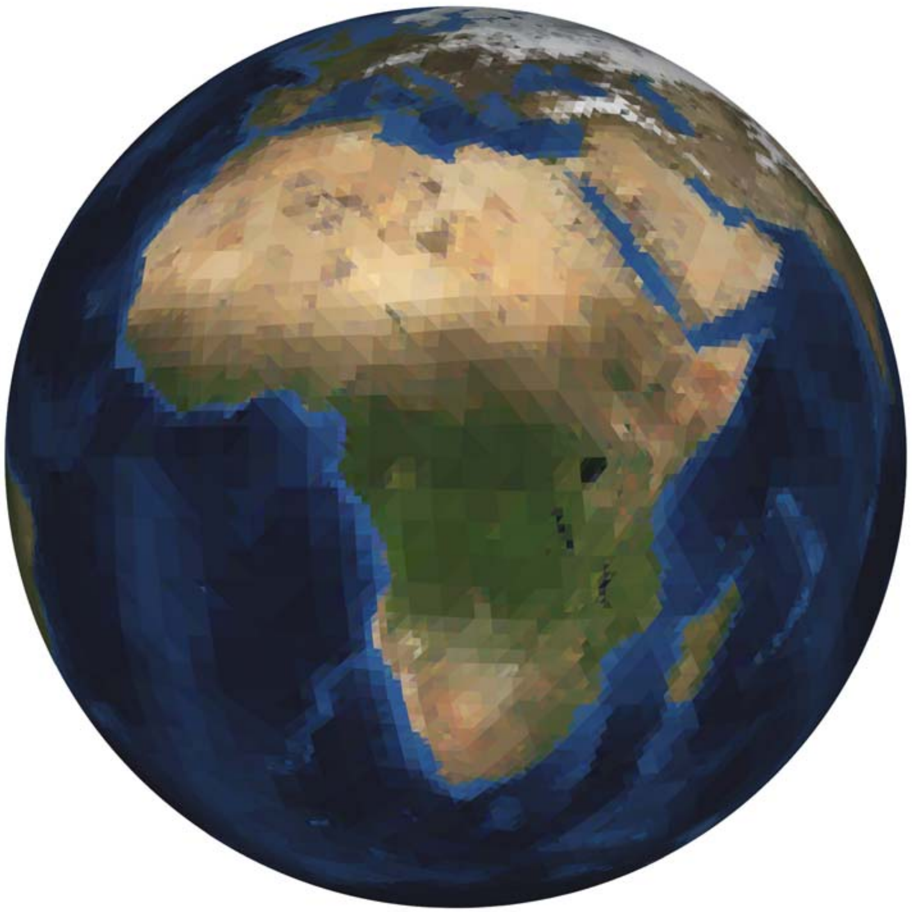
Experiments

- Number of basis function coefficients: 8192



Experiments

- Number of basis function coefficients: 16384



Experiments

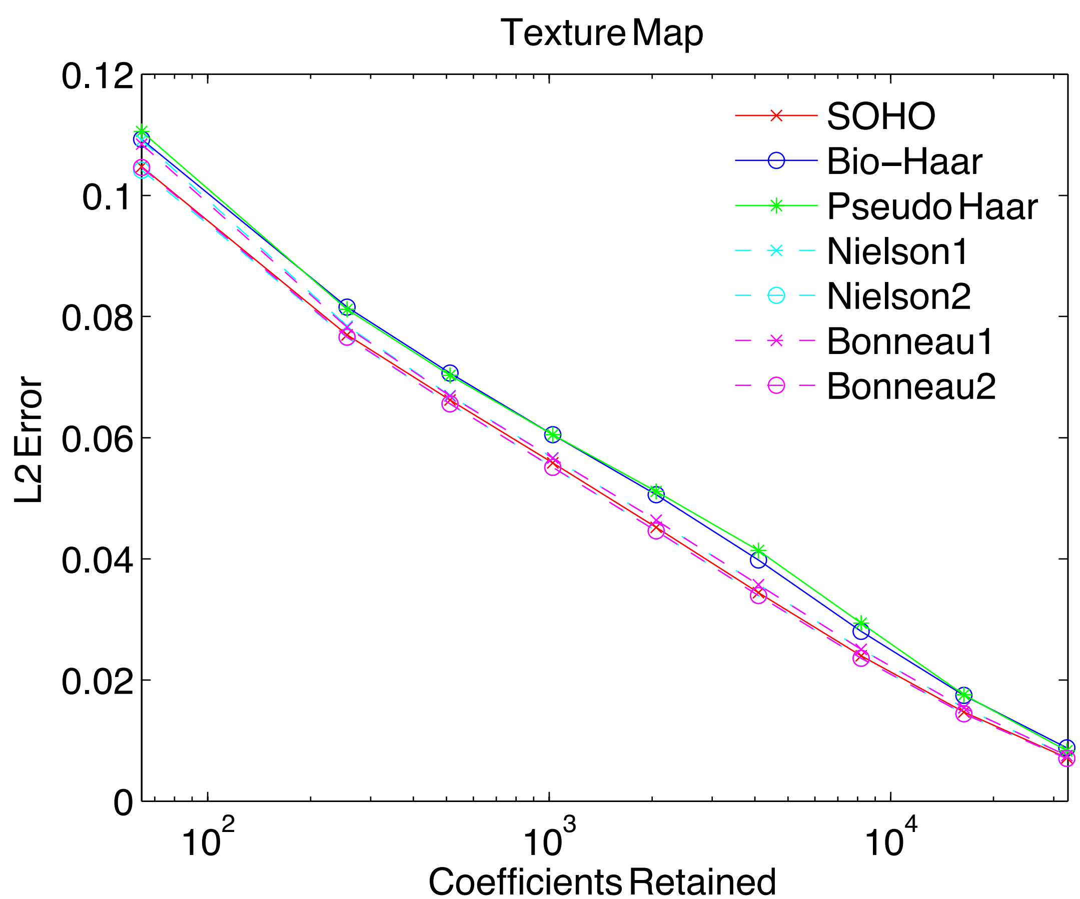
- Number of basis function coefficients: 32768

Experiments

- Number of basis function coefficients: all

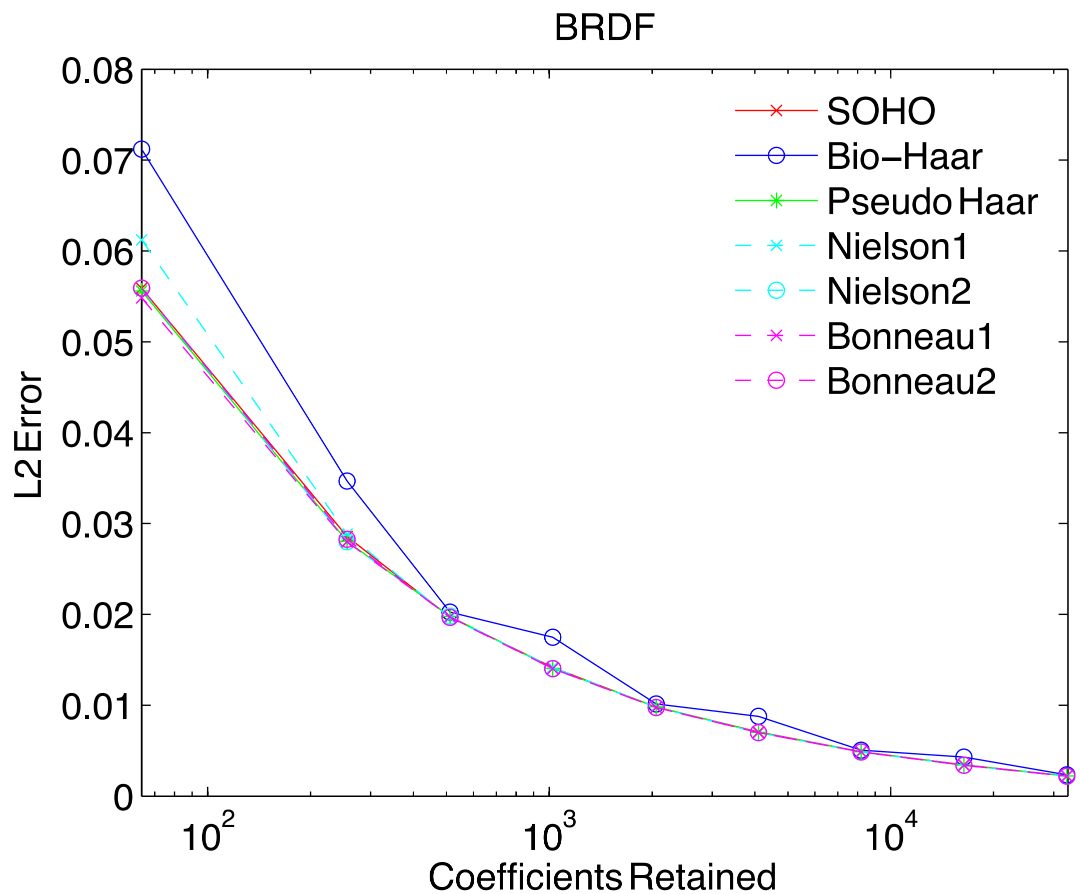
Experiments

- Approximation of texture map



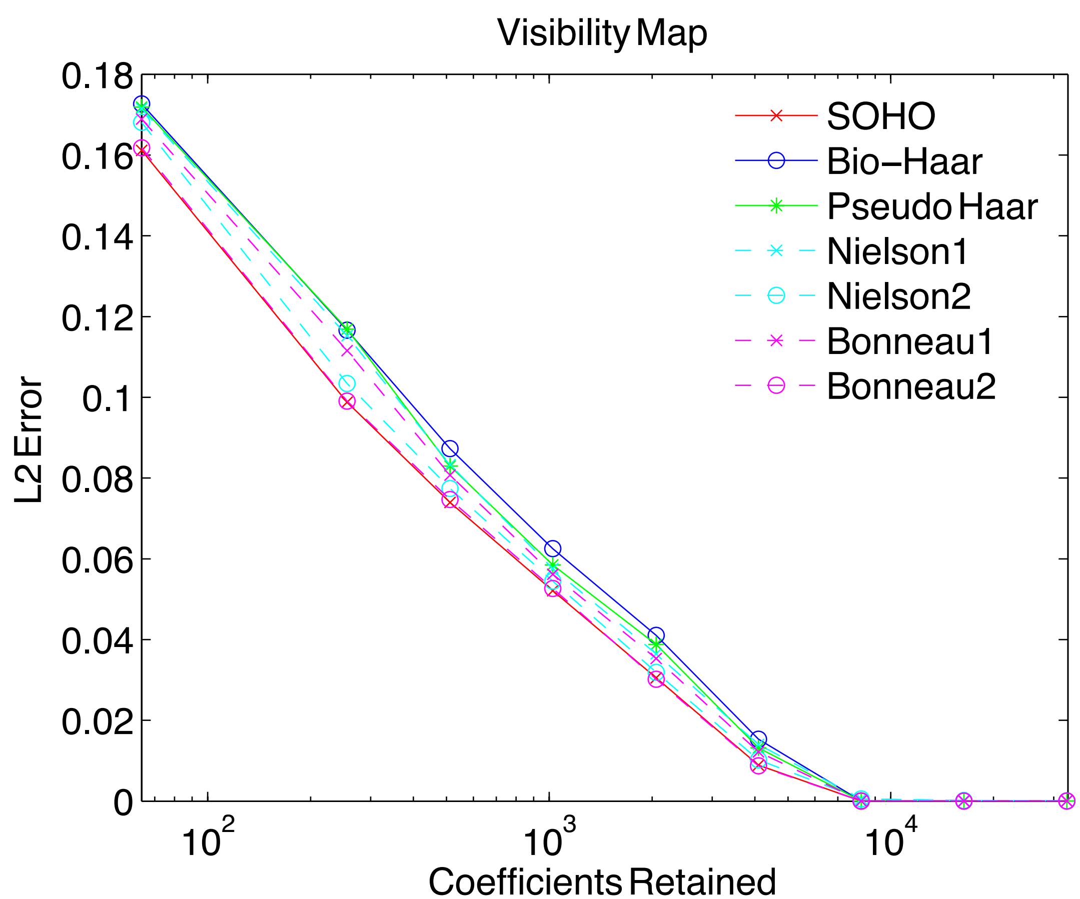
Experiments

- Approximation of BRDF



Experiments

- Approximation of visibility map



Rotation of Signals

- Rotation and alignment of signals necessary in many applications

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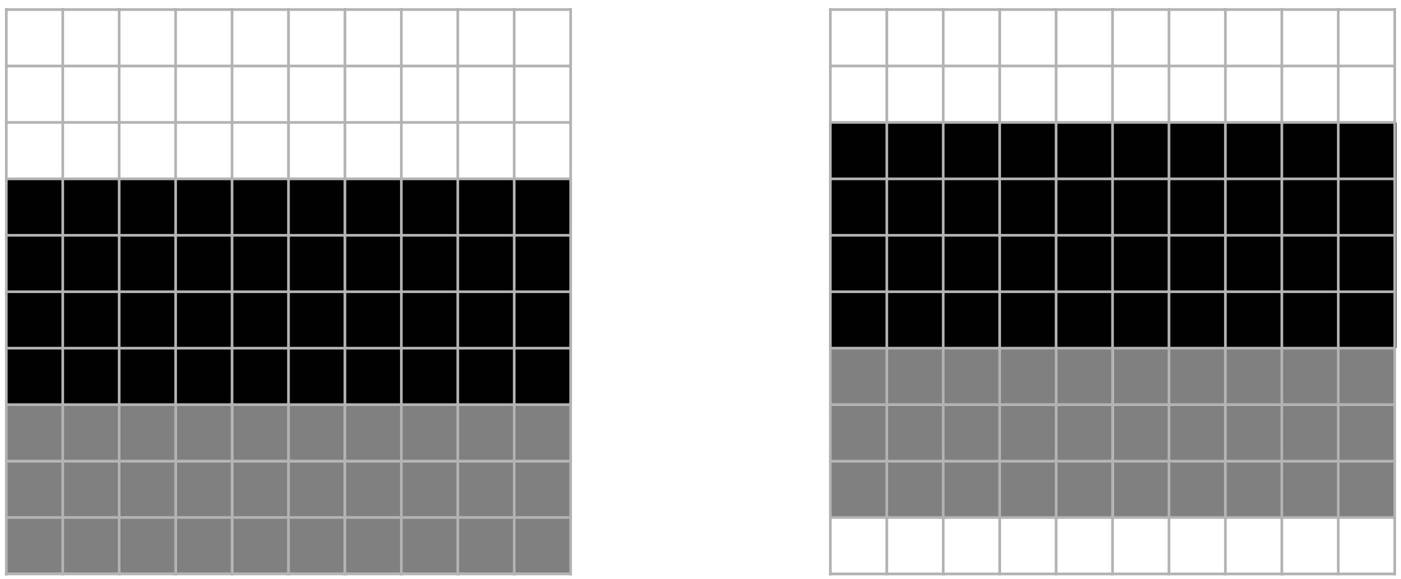

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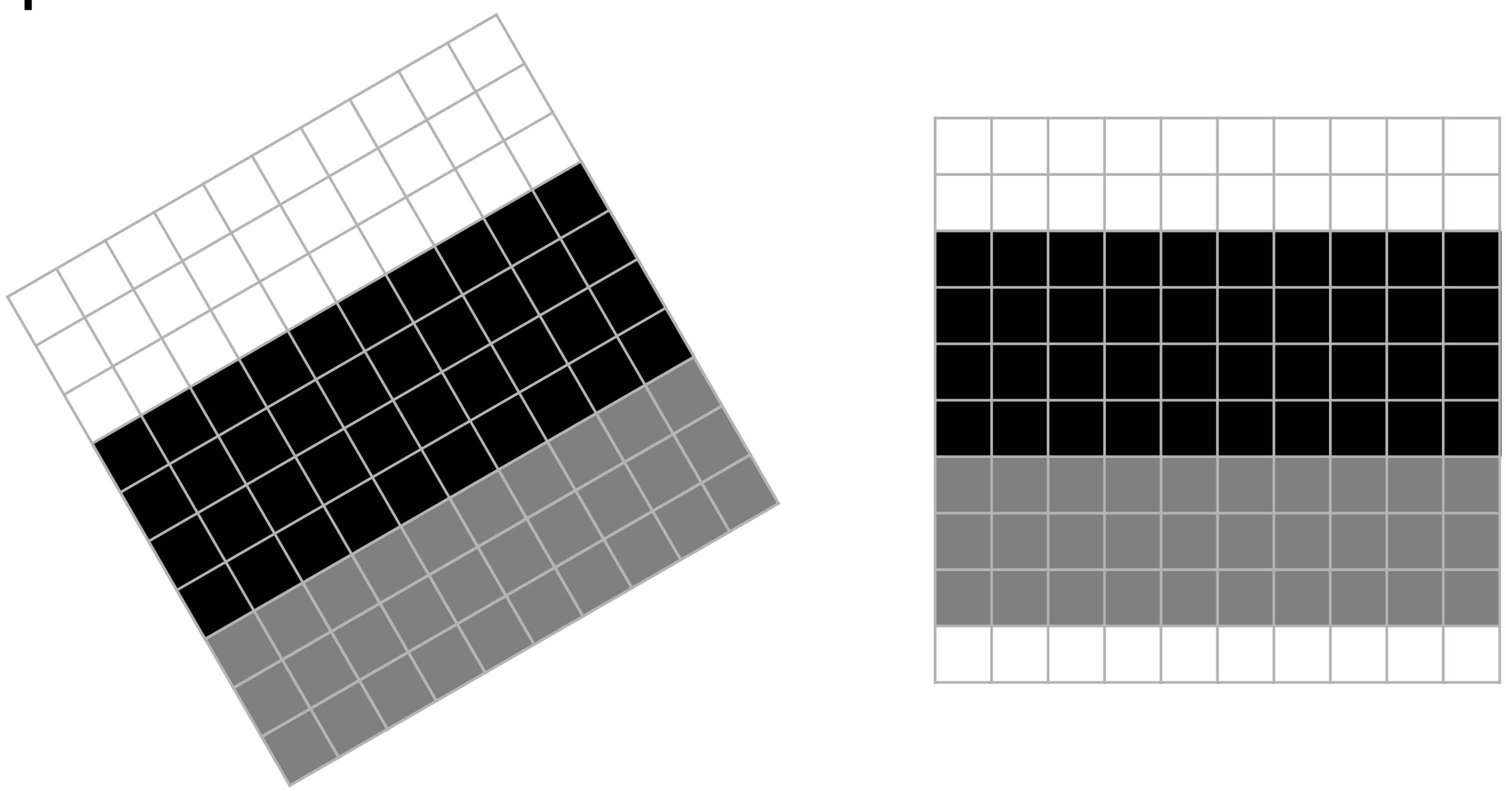

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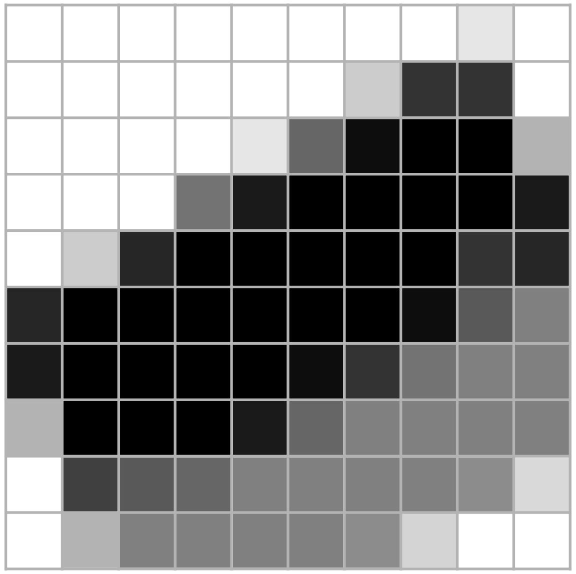
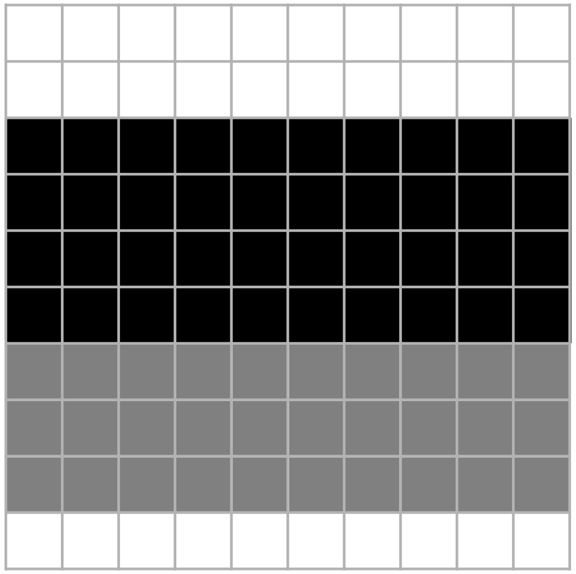
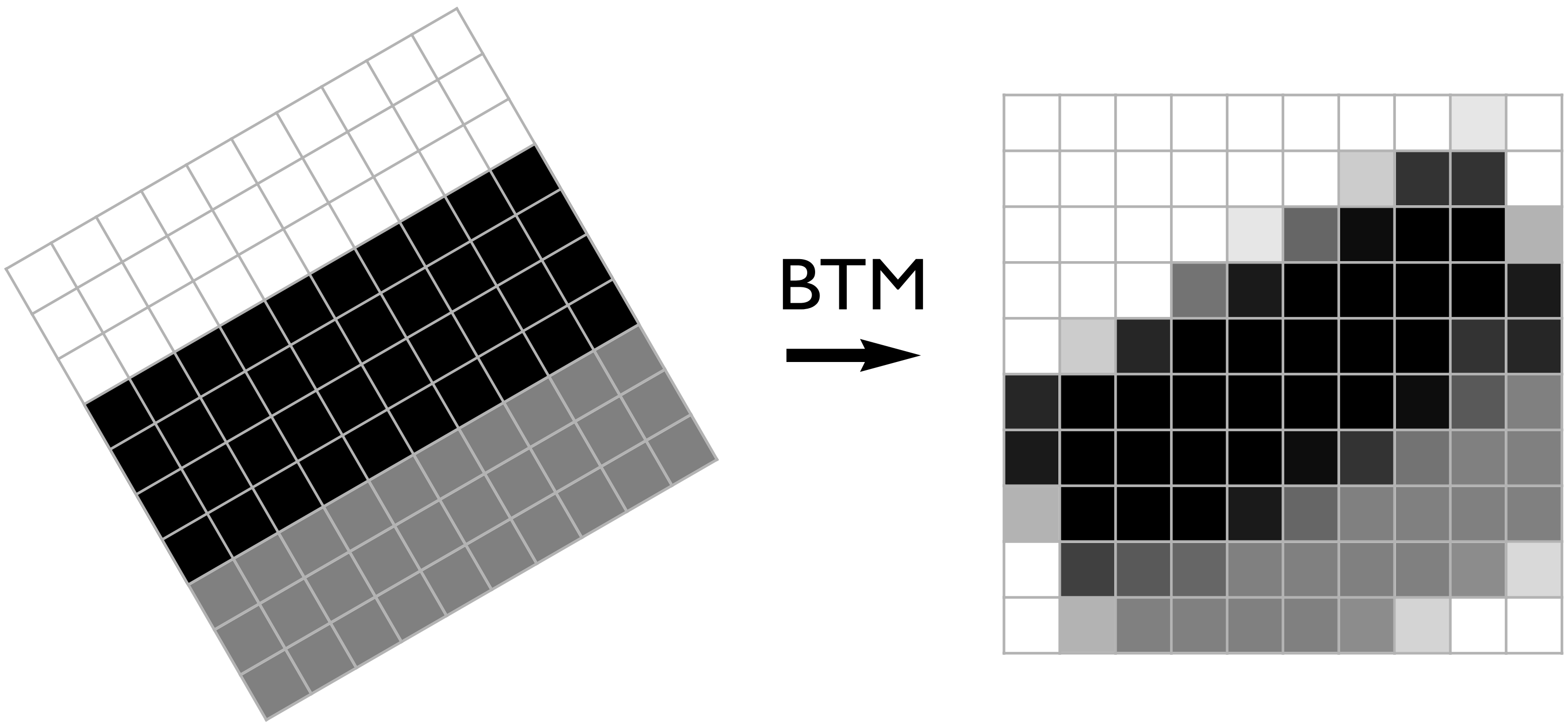




Rotation of Signals

- *Basis transformation matrix* allows to project from the rotated source basis into the unrotated target basis



Source Basis

Target Basis

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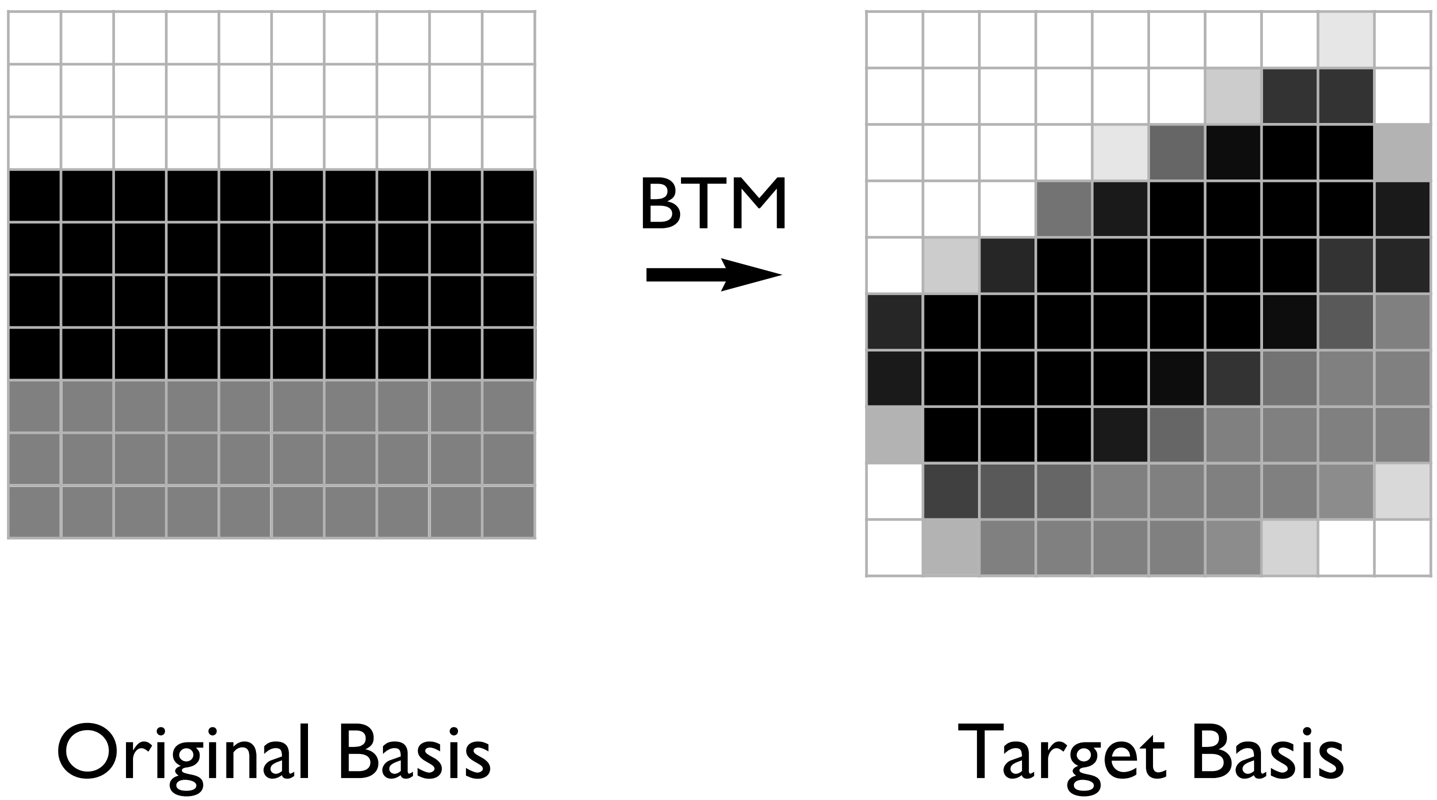

Rotation of Signals

- Previous work: Planar representations of spherical signals [Wang 2006]

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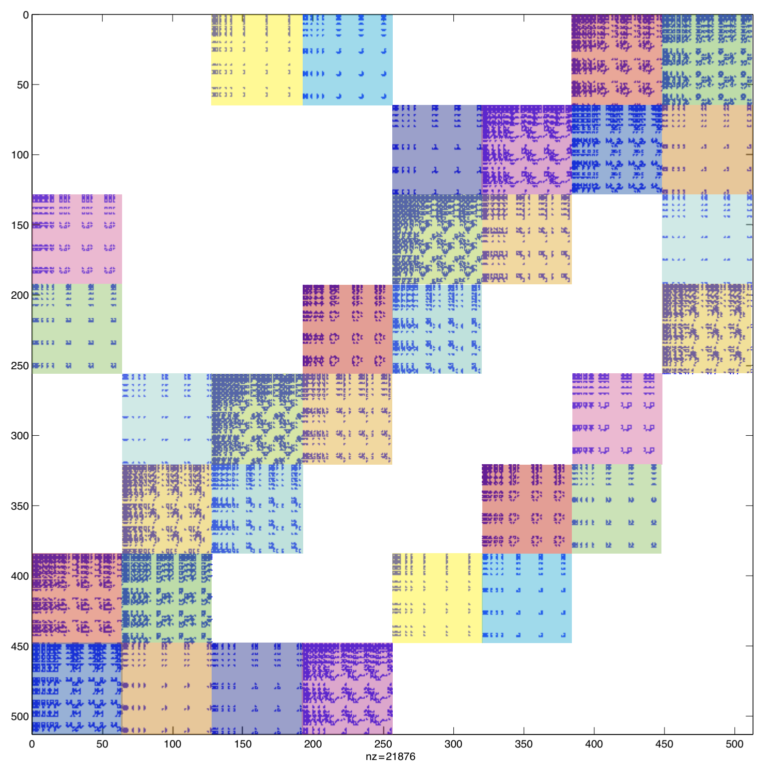
⇒ Numerical computation of basis transformation matrix elements

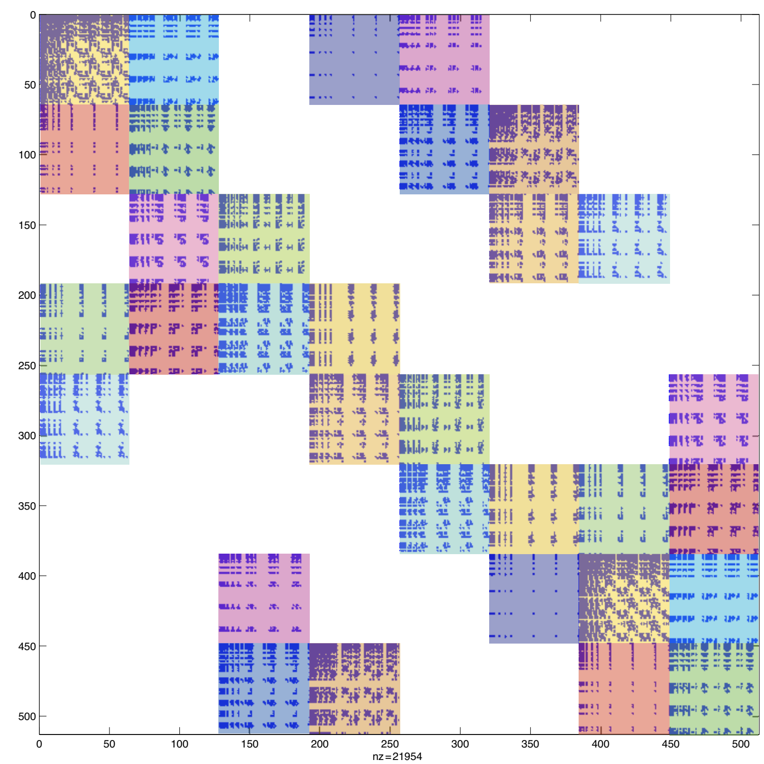
- Spherical Haar wavelet bases: Rotation amounts to translation of basis functions on the sphere

⇒ Analytic computation of matrix elements

Rotation of Signals

- Rotation matrices for spherical Haar wavelet bases have *quasi block symmetric* structure





Rotation of Signals

- Experiments

Source Basis

Target Basis

Conclusion

- SOHO wavelet basis: orthogonal and symmetric spherical Haar wavelet basis
- Well suited for the efficient approximation and processing of spherical signals
- Competitive or lower error rates than previously proposed spherical Haar wavelet bases for the approximation of all-frequency signals
- Signals represented in the SOHO wavelet basis can be rotated with basis transformation matrices: analytic computation of the matrix elements possible

Outlook

- Operation-aware representations
  - Design of representations for specific applications and operations
  - Requires detailed understanding of representations and *how* these interact with operations
  - Partly studied in physics and applied mathematics but often ignored in other fields

Precomputed Radiance Transfer

- Challenges
  - Dynamic Scenes
  - Multiple bounces

Precomputed Radiance Transfer

- Challenges
  - Dynamic Scenes
  - Multiple bounces
- Looking for algorithms which are
  - sparse
  - predictable
  - cheap

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