The Area Formulation of Light Transport

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Abstract

The area formulation of the light transport equation is frequently employed for theoretical investigations in computer graphics, and in fact it was this form that was first introduced by Kajiya. However, in the literature currently only heuristic justifications for the necessary change of variables from the more common angular formulation exist. In this note, we will present a rigorous derivation of the change of variables to the area formulation which also clarifies its intrinsic structure.

In the area formulation of the light transport equation, the solid angle measure $d\omega = \sin\theta \, d\theta \, d\phi$[1] is replaced by the surface area measure dA for a surface subtending the solid angle $d\omega$, cf. Fig. 0.1. In classical notation, the change of variables is given by

$$d\omega = \frac{\cos\bar{\theta}}{\|\mathbf{x} - \bar{\mathbf{x}}\|^2} dA \tag{1}$$

where we omitted momentarily the visibility term which will naturally arise again later.

Currently, no rigorous derivation of the change of variables in Eq. 1 exists in the computer graphics literature.[2] In the following, we will fill this gap and we will obtain the area formulation by studying the exterior 2-form

$$\beta(p) = \frac{x \, dy \wedge dy + y \, dz \wedge dx + z \, dx \wedge dy}{(x^2 + y^2 + z^2)^{3/2}} \in \Omega^2(\mathbb{R}^3 \backslash \{0\}) \tag{2}$$

where $p = (x, y, z) \in \mathbb{R}^3$.[3] Intuitively, one should think of the 2-form β as being "centered" at the coordinate origin chosen to be $\mathbf{x} \in \mathbb{R}^3$ with $p = \bar{\mathbf{x}}$, see again Fig. 0.1.

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[1]We cannot refrain from re-iterating that, despite the common notation, $d\omega$ is not an exact 2-form.

[2]For discussions of the area formulation in the computer graphics literature see for example (Dutré, Bala, and Bekaert, *Advanced Global Illumination*, p. 24) or (Pharr and Humphreys, *Physically Based Rendering: From Theory to Implementation*, Chapter 5.5.3).

[3]For the necessary background on differential forms see for example (Frankel, *The Geometry of Physics*).

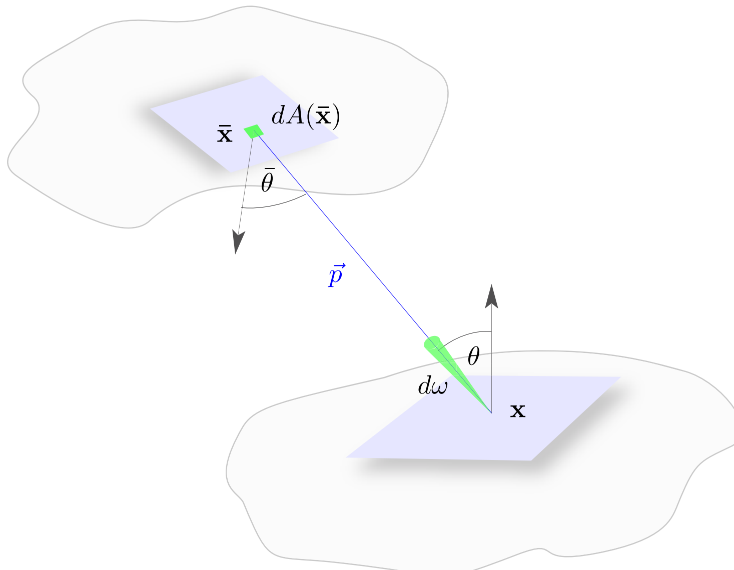

Figure 0.1: Geometry of the area formulation.

We will begin our derivation by establishing that the 2-form β in Eq. 2 is closed, then will study its pullback onto an arbitrary surface $\mathcal{M} \subset \mathbb{R}^3$, and using these results we will finally show that the integral of $\beta \in \Omega^2(\mathbb{R}^3 \setminus \{0\})$ over $\mathcal{M}$ equals the solid angle subtended by the surface $\mathcal{M}$, providing the desired result.[4] Although the following calculations are rather lengthy, these are standard operations of exterior calculus in $\mathbb{R}^3$ and much of their length results from the verbosity we have chosen to provide.

The 2-form β is closed. Computing the exterior derivative $\mathrm{d}\beta \in \Omega^3(\mathbb{R}^3 \setminus \{0\})$, we obtain for the first term that

$$\mathrm{d}\left(\frac{x}{(x^2+y^2+z^2)^{3/2}}\right) dy \wedge dz = \frac{\partial}{\partial x}\left(\frac{x}{(x^2+y^2+z^2)^{3/2}}\right) dx \wedge dy \wedge dz \tag{3a}$$

where the remaining partial derivatives vanish by the anti-symmetry of the wedge product that implies $dy \wedge dy = 0$ and $dz \wedge dz = 0$. Using the product rule we obtain

$$\mathrm{d}\left(\frac{x}{(x^2+y^2+z^2)^{3/2}}\right) dy \wedge dz \tag{3b}$$

$$= \frac{(x^2+y^2+z^2)^{3/2} - \frac{3}{2}x\,(x^2+y^2+z^2)^{1/2}\,2x}{(x^2+y^2+z^2)^{6/2}} dx \wedge dy \wedge dz \tag{3c}$$

[4]Our treatment was inspired and motivated by (Spivak, *Calculus on Manifolds: A Modern Approach to Classical Theorems of Advanced Calculus*, Exercise 5-31).

$$= \frac{(x^2+y^2+z^2)^{1/2}\left((x^2+y^2+z^2)-3x^2\right)}{(x^2+y^2+z^2)^{6/2}}\,dx\wedge dy\wedge dz \tag{3d}$$

and hence we have

$$\mathrm{d}\left(\frac{x}{(x^2+y^2+z^2)^{3/2}}\right)dy\wedge dz = \frac{-2x^2+y^2+z^2}{(x^2+y^2+z^2)^{5/2}}\,dx\wedge dy\wedge dz. \tag{3e}$$

Analogously, for the other two terms arising in $\mathrm{d}\beta$ we have

$$\mathrm{d}\left(\frac{y}{(x^2+y^2+z^2)^{3/2}}\right)dz\wedge dx = \frac{x^2-2y^2+z^2}{(x^2+y^2+z^2)^{5/2}}\,dx\wedge dy\wedge dz \tag{3f}$$

and

$$\mathrm{d}\left(\frac{z}{(x^2+y^2+z^2)^{3/2}}\right)dx\wedge dy = \frac{x^2+y^2-2z^2}{(x^2+y^2+z^2)^{5/2}}\,dx\wedge dy\wedge dz. \tag{3g}$$

The exterior derivative $\mathrm{d}\beta \in \Omega^3(\mathbb{R}^3\setminus\{0\})$ is hence given by

$$\mathrm{d}\left(\frac{x\,dy\wedge dz + y\,dz\wedge dz + z\,dx\wedge dy}{(x^2+y^2+z^2)^{3/2}}\right) \tag{4a}$$

$$= \frac{-2x^2+y^2+z^2+x^2-2y^2+z^2+x^2+y^2-2z^2}{(x^2+y^2+z^2)^{5/2}}\,dx\wedge dy\wedge dz \tag{4b}$$

$$= \frac{0}{(x^2+y^2+z^2)^{5/2}}\,dx\wedge dy\wedge dz \tag{4c}$$

which shows that β is closed, that is $\mathrm{d}\beta = 0$.

The pullback of β onto an arbitrary surface. The 2-form β obtains physical significance when it is integrated over a surface $\mathcal{M}\subset\mathbb{R}^3$, which requires to pull it back onto $\mathcal{M}$ and then into an appropriate chart for the surface. This is conveniently computed as

$$(\varphi^*\beta_p)(u,v) = \beta_p\left(\frac{\partial}{\partial u},\frac{\partial}{\partial v}\right)du\,dv \tag{5}$$

where $\partial/\partial u$ and $\partial/\partial v$ are the tangent vectors in $T_p\mathcal{M}$ induced from a chart for $\mathcal{M}$ with coordinates (u,v). Writing for convenience $\|p\|^3 = (x^2+y^2+z^2)^{3/2}$ and since $(\alpha\wedge\beta)(v,w) = \alpha(v)\,\beta(w) - \alpha(w)\beta(v)$ for arbitrary 1-forms α,β and vectors v,w, we obtain for the first term of the above pullback

$$\left(\frac{x}{\|\vec{p}\|^3}dy\wedge dz\right)\left(\frac{\partial}{\partial u},\frac{\partial}{\partial v}\right) = \frac{x}{\|\vec{p}\|^3}\,(dy\wedge dz)\left(\frac{\partial}{\partial u},\frac{\partial}{\partial v}\right) \tag{6a}$$

$$= \frac{x}{\|\vec{p}\|^3}\,dy\left(\frac{\partial}{\partial u}\right)dz\left(\frac{\partial}{\partial v}\right) - dy\left(\frac{\partial}{\partial v}\right)dz\left(\frac{\partial}{\partial u}\right) \tag{6b}$$

and by the biorthogonality of the basis functions this is equivalent to

$$\left(\frac{x}{\|\vec{p}\|^3} dy \wedge dz\right)\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right) = \frac{x}{\|\vec{p}\|^3} \frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial y}{\partial v}\frac{\partial z}{\partial u}. \tag{6c}$$

Analogously, we obtain for the other two terms arising from the pullback that

$$\left(\frac{y}{\|\vec{p}\|^3} dz \wedge dx\right)\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right) = \frac{y}{\|\vec{p}\|^3} \frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial z}{\partial v}\frac{\partial x}{\partial u} \tag{6d}$$

and

$$\left(\frac{z}{\|\vec{p}\|^3} dx \wedge dy\right)\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right) = \frac{z}{\|\vec{p}\|^3} \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}. \tag{6e}$$

The partial derivatives in Eqs. 6c–6e are the component form of the cross product, and hence, for an appropriately chosen orientation, the pullback in Eq. 5 can be written as

$$(\varphi^* \beta_p)(u,v) = \frac{\vec{p} \cdot \vec{n}(p)}{\|\vec{p}\|^3}\, du\, dv \tag{7a}$$

where $\vec{n}(p)$ is the local surface normal for $\mathcal{M}$ at p. Writing $\vec{n}(p)$ as $\vec{n}(p) = \|\vec{n}\|\bar{n}$, where $\bar{n}$ is a unit vector, this is equivalent to

$$(\varphi^* \beta_p)(u,v) = \frac{\vec{p} \cdot \bar{n}(p)}{\|\vec{p}\|^3}\, \|\vec{n}\|\, du\, dv \tag{7b}$$

$$= \frac{\vec{p} \cdot \bar{n}(p)}{\|\vec{p}\|^3}\, dA \tag{7c}$$

where we also used that, by definition, $dA = \|\vec{n}\|\, du\, dv$. With an analogous decomposition of $\vec{p}$ as $\vec{p} = \|p\|\bar{p}$ we obtain

$$(\varphi^* \beta_p)(u,v) = \frac{\|\vec{p}\|}{\|\vec{p}\|^3}\, \bar{p} \cdot \bar{n}(p)\, dA \tag{7d}$$

$$= \frac{\cos \bar{\theta}}{\|\vec{p}\|^2}\, dA \tag{7e}$$

where $\cos\bar{\theta} = \bar{p} \cdot \bar{n}(p)$. The last equations recovers the right hand side of Eq. 1.

The area formulation With the foregoing results, the left hand side of Eq. 1 is obtained by considering a volume N as shown in Fig. 0.2 where $\partial N_1 \subset S^2$ is the unique image of ∂N_3 under the inverse time evolution diffeomorphism of light transport $\eta_{-t} : \mathbb{R} \times T^*Q \to T^*Q$,[5] and ∂N_3 is the visible part of $\mathcal{M}$ as

[5] Cf. (Lessig, "Modern Foundations of Light Transport Simulation", Chapter 3). Note that the uniqueness of the image under time evolution is in general not guaranteed.

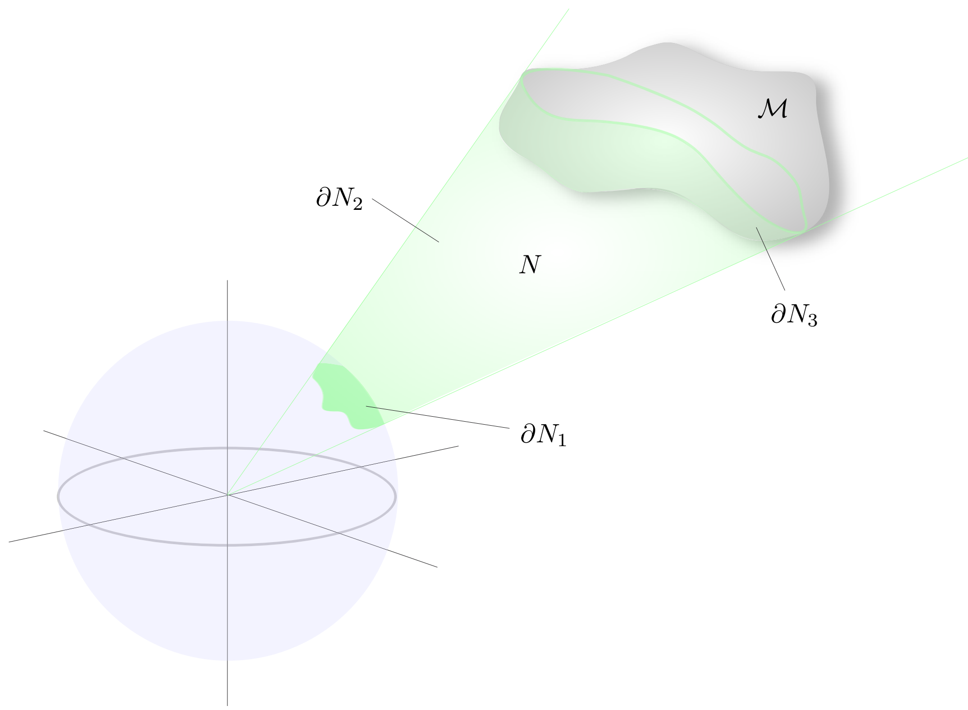

Figure 0.2: The solid angle ∂N_1 subtended by $\mathcal{M}$ is given by the surface integral of the 2-form in Eq. 2 over the visible part of the surface.

seen from the origin, which in the computer graphics literature is commonly expressed using the binary visibility function. Since β is closed we have

$$0 = \int_N \mathrm{d}\beta \tag{8a}$$

and by Stoke's theorem this is equivalent to

$$0 = \int_{\partial N} \beta. \tag{8b}$$

The boundary ∂N consists of three parts and hence we have for the surface integral

$$0 = \int_{\partial N_1} \beta + \int_{\partial N_2} \beta + \int_{\partial N_3} \beta. \tag{8c}$$

The integral over ∂N_2 vanishes since there the cosine term vanishes. Hence, we obtain

$$\int_{\partial N_1} \beta = - \int_{\partial N_3} \beta \tag{8d}$$

which shows that the surface integral of β over an arbitrary manifold equals the solid angle subtended by the manifold, and since N is arbitrary the equality also holds infinitesimally. Together with Eq. 7e this justifies Eq. 1.

Bibliography

Dutré, P., K. Bala, and P. Bekaert. *Advanced Global Illumination*. Natick, MA, USA: A. K. Peters, Ltd., 2006.

Frankel, T. *The Geometry of Physics*. Cambridge University Press, 2003.

Lessig, C. "Modern Foundations of Light Transport Simulation". Ph.D. thesis. Toronto: University of Toronto, 2012. URL: http://www.dgp.toronto.edu/people/lessig/dissertation/.

Pharr, M. and G. Humphreys. *Physically Based Rendering: From Theory to Implementation*. second ed. San Francisco, CA, USA: Morgan Kaufmann Publishers Inc., 2010.

Spivak, M. *Calculus on Manifolds: A Modern Approach to Classical Theorems of Advanced Calculus*. HarperCollins Publishers, 1971.