CSC 418 Winter 2007 midterm solutions

(1) a) examine the slope of $f(x) = x^2/2a$

if $f' > 1$ on the interval, iterate on y
if $f' < 1$ on the interval, iterate on x
2 or 0
-1 for errors

($f' = 0$ is arbitrary, go with some convention)

b) $f(x,y) = 0$ defines the curve
$f(x,y) > 0$ is one side
$f(x,y) < 0$ is the other side

observing the sign of $f(x,y)$ determines which side to draw on

so $E = f(x,y)$

$f(x,y) = y - \frac{x^2}{2a} = E_{x,y}$

$E_{x,y+1} = f(x,y+1) = y+1 - \frac{x^2}{2a}$

$E_{x+1,y+1} = f(x+1,y+1) = y+1 - \frac{x^2}{2a} - \frac{(2x+1)}{2a}$

$E_{x+1,y} = f(x+1,y) = y - \frac{x^2}{2a} - \frac{(2x+1)}{2a}$

since we're iterating on y, we only need $E_{x,y+1}$ and $E_{x+1,y+1}$

To get $E_{x+1,y+1}$ from $E_{x,y}$ (and $E_{x,y+1}$ from $E_{x,y}$)
examine their difference; add the difference to $E_{x,y}$ to get the updated error

$x,y+1$ update: $E_{x,y+1} - E_{x,y} = \left(y+1-\frac{x^2}{2a}\right) - \left(y - \frac{x^2}{2a}\right) = 1$

$x+1,y+1$ update: $E_{x+1,y+1} - E_{x,y} = \left(y+1-\frac{x^2}{2a} - \frac{(2x+1)}{2a}\right) - \left(y - \frac{x^2}{2a}\right) = 1 - \frac{2x+1}{2a}$

$x+1,y$ update: $E_{x+1,y} - E_{x,y} = \left(y - \frac{x^2}{2a} - \frac{(2x+1)}{2a}\right) - \left(y - \frac{x^2}{2a}\right) = -\frac{2x+1}{2a}$

the updates are in fp math because of the $\frac{1}{2a}$ factor, so multiply everything by $2a$ to gain integer versions

$\Delta_{x,y+1} = 2a$

$\Delta_{x+1,y+1} = 2a - 2x - 1$

1 for what E is
2 for the update
1 for integer versions

having $f(x,y) = \frac{x^2}{2a} - y$ does not affect anything so may have updates

$\Delta_{x,y+1} = -2a$
$\Delta_{x+1,y+1} = 1 + 2x - 2a$

instead

c) no; for $x \in [2a, 4a]$ (original interval), $f'(x) > 1$ so iterating on y is used

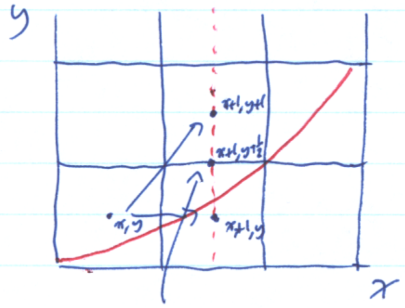
1 for $x \in [0, a]$ (part of new interval), $f'(x) < 1$, so need to iterate on x

So, separate ~~into~~ the parabola into two segments, and iterate over x where $f' < 1$ and iterate over y where $f' > 1$.

~~Additionally, the error to maintain is different~~
If iterating over x, the ~~up~~ error update $\Delta_{x+1,y+1}$ needs to be incorporated, and $\Delta_{x,y+1}$ won't be used in that case.

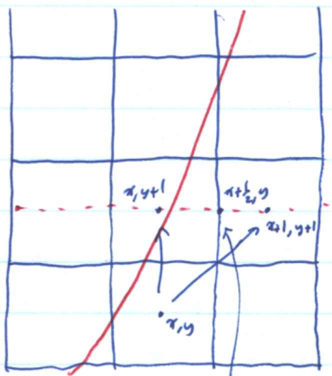
Additionally, the error to be maintained is different between iterating over x and over y:

While iterating over x, one is looking at the midpoint between y coords while when iterating over y, one is looking at the midpoint between x coords



testing midpoint to see which side of the parabola it lies on:

$f(x+1, y+\frac{1}{2}) < 0$ or > 0



testing midpoint to see which side of the parabola it lies on:

$f(x+\frac{1}{2}, y+1) < 0$ or > 0

This only matters to the initialization since difference between midpoints to check remains 1 or 0 in the coords

~~4 or 0~~ 4, 2, 0 for material in block ①

2 for nothing about slope being the reason

Notes about Q 1:

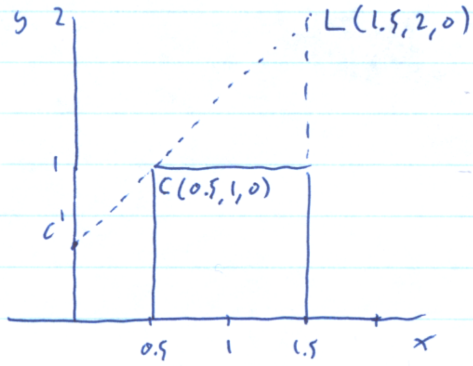
the point $(2a, 2a)$ does not lie on the parabola; if the function was

$$y - \left(\frac{x}{2a}\right)^2 = 0$$

then that would have been the case.

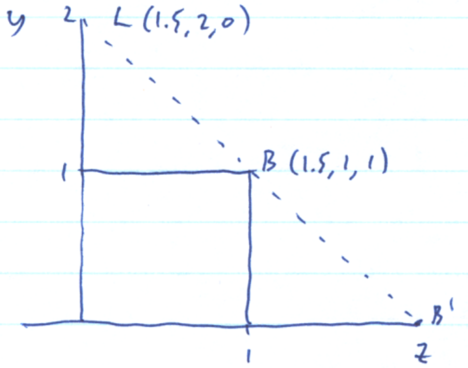
~~Any points~~ There was no penalty when discussing $(2a, 2a)$ as if it were the slope's turning point

2 a)

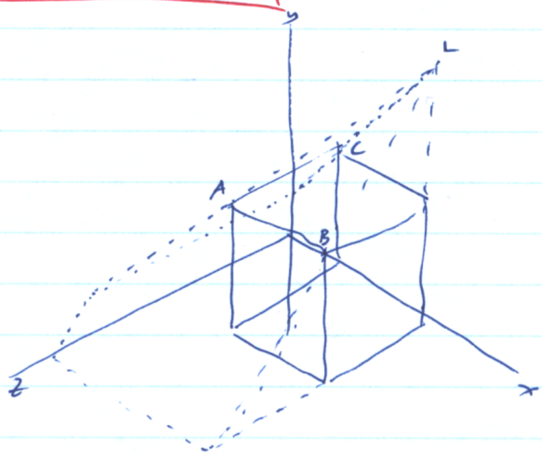


$$\boxed{C' = (0, 0.5, 0)}$$

1 mark per correct point
0 if wrong and without work
½ if wrong with correct work



$$\boxed{B' = (1.5, 0, 2)}$$



the ray through A hits the yz plane ($x=0$), or at least appears to

try $x=0$, see if the point makes sense

$\vec{L} = (1.5, 2, 0)$ $\quad\quad$ $\vec{A} = (0.5, 1, 1)$

ray: $\vec{x}(\lambda) = (\vec{A} - \vec{L})\lambda + \vec{L}$

$$= \begin{bmatrix} 0.5-1.5 \\ 1-2 \\ 1-0 \end{bmatrix} \lambda + \begin{bmatrix} 1.5 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \lambda + \begin{bmatrix} 1.5 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ ? \\ ? \end{bmatrix}$$

$$-1 \cdot \lambda + 1.5 = 0$$
$$\lambda = 1.5$$

so $A' = \begin{bmatrix} 1.5 \\ 2 \\ 0 \end{bmatrix} + 1.5 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2-1.5 \\ 0+1.5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.5 \\ 1.5 \end{bmatrix}$

$$\boxed{A' = (0, 0.5, 1.5)}$$

the point lies in the right octant ($x \geq 0, y \geq 0, z \geq 0$) so it's good

(2) b) take $\vec{P}$, construct the ray through $\vec{L}$ and P, originating from $\vec{P}$

$$\vec{x}(\lambda) = (\vec{P} - \vec{L})\lambda + \vec{P}$$

intersect the ray with the three planes, $x=0$, $y=0$, $z=0$
($\vec{n} = \hat{i}$, $\vec{n} = \hat{j}$, $\vec{n} = \hat{k}$)

solve for λ and check λ and $\vec{x}(\lambda)$ against restrictions:

$$\underbrace{\lambda \geq 0}_{\text{forward ray direction}}, \quad \underbrace{x \geq 0, y \geq 0, z \geq 0}_{\text{correct octant}}$$

using the implicit plane equation, $(\vec{x} - \vec{x}_0) \cdot \vec{n} = 0$, we can find λ appropriately.
($\vec{n}$ is ~~one of~~ chosen to correspond to the plane to intersect with, $\vec{x}_0 = \vec{0}$ since all planes we want go through the origin)

$$\vec{x}(\lambda) \cdot \vec{n} = 0$$

$$((\vec{P} - \vec{L})\lambda + \vec{P}) \cdot \vec{n} = 0$$

$$\lambda = \frac{-\vec{P} \cdot \vec{n}}{(\vec{P} - \vec{L}) \cdot \vec{n}}$$

so $\vec{x}(\lambda) = (\vec{P} - \vec{L}) \dfrac{-\vec{P} \cdot \vec{n}}{(\vec{P} - \vec{L}) \cdot \vec{n}} + \vec{P}$

if λ dne (ie $(\vec{P} - \vec{L}) \cdot \vec{n} = 0$) the ray is perpendicular to the normal, so either the line runs parallel to the plane, never touching, or it is embedded in the plane. Test $\vec{P}$ to find out

1 plane-ray intersection idea
1 math/discussion on how to get to finding P'
0.5 for the point P'
1 for $\lambda \geq 0$, $x \geq 0$, $y \geq 0$, $z \geq 0$
0.5 for λ dne case

*alternate ~~partial~~ ~~partist~~ solution

$P(x,y,z)$ $\quad L=(1.5,2,0)$

the ray through $\vec{P}$ and $\vec{L}$, originating at ~~P~~ $\vec{L}$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} x-1.5 \\ y-2 \\ z-0 \end{bmatrix}\lambda + \begin{bmatrix} 1.5 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} x-1.5 \\ y-2 \\ z \end{bmatrix}\lambda + \begin{bmatrix} 1.5 \\ 2 \\ 0 \end{bmatrix}$$

we can write this in symmetric form

$$\left.\begin{aligned} x' &= (x-1.5)\lambda+1.5 \\ y' &= (y-2)\lambda+2 \\ z' &= z\lambda \end{aligned}\right\} \rightarrow \quad \lambda = \frac{x'-1.5}{x-1.5} = \frac{y'-2}{y-2} = \frac{z'}{z}$$

test planes $x=0$, $y=0$, but not $z=0$ as its impossible to hit that plane directly

$x'=0$: $\quad \dfrac{y'-2}{y-2} = \dfrac{0-1.5}{x-1.5} \qquad \dfrac{z'}{z} = \dfrac{0-1.5}{x-1.5}$

$$y' = -1.5\frac{y-2}{x-1.5}+2 \qquad z' = z\frac{-1.5}{x-1.5}$$

$y'=0$: $\quad \dfrac{x'-1.5}{x-1.5} = \dfrac{0-2}{y-2} \qquad \dfrac{z'}{z} = \dfrac{0-2}{y-2}$

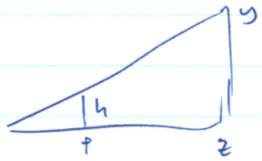
$$x' = -2\frac{x-1.5}{y-2}+1.5 \qquad z' = -2\frac{z}{y-2}$$

to sort out which point intersection is the right one, one can check to see if $P'(x',y',z')$ lies in the correct ~~qu~~ octant. However, that does not eliminate backwards running rays. To check this, simply compute λ for one of the ratios

What happens if one of the denominators is zero?
The denominator is a component of the line's direction vector, so if it's zero, the corresponding coordinate is invariant, and that term may be excluded from computation.

1 plane-ray intersection
1 math-discussion
(+1 included if wrong $x=0, y=0$ bounds)
-0.5 if does not discuss → case
0.5 point p'
0.5 → dne case (denominator $= 0$ case)

② c) Consider perspective projection of an object onto an image plane



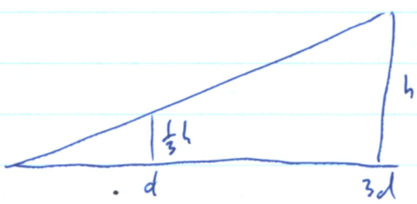
unknown h is related to everything else by

$$\frac{h}{f} = \frac{y}{z}$$

$$h = f\frac{y}{z}$$

so it only depends on the distance / height ratio.

So of course, an object 3 times as far and one $\frac{1}{3}$ as big appear to have the same size on the image plane as they have the same distance / height ratio



1 idea
1 illustrative picture

2d) Consider a line in world space

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} + \lambda \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}$$

under perspective projection;

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \to \begin{bmatrix} \frac{x}{z} f \\ \frac{y}{z} f \end{bmatrix} \qquad f \text{ some constant}$$

so
$$\begin{bmatrix} a_x + \lambda b_x \\ a_y + \lambda b_y \\ a_z + \lambda b_z \end{bmatrix} \to f \begin{bmatrix} \frac{a_x + \lambda b_x}{a_z + \lambda b_z} \\ \frac{a_y + \lambda b_y}{a_z + \lambda b_z} \end{bmatrix}$$

as $\lambda \to \infty$ (or $-\infty$ depending on orientation of the line);

$$\lim_{\lambda \to \infty} f \begin{bmatrix} \frac{a_x + \lambda b_x}{a_z + \lambda b_z} \\ \frac{a_y + \lambda b_y}{a_z + \lambda b_z} \end{bmatrix} = \begin{bmatrix} f \frac{b_x}{b_z} \\ f \frac{b_y}{b_z} \end{bmatrix}$$

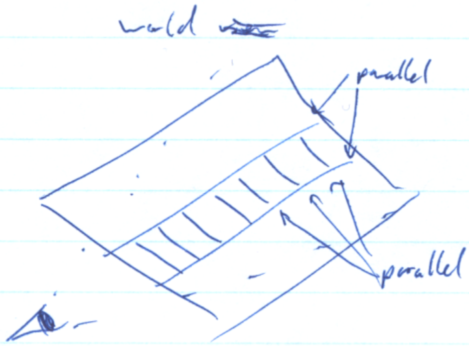
so it doesn't matter which point the lines go through; as long as they have the same direction vector, they converge to the same ~~spot~~ point, as long as the above are defined.

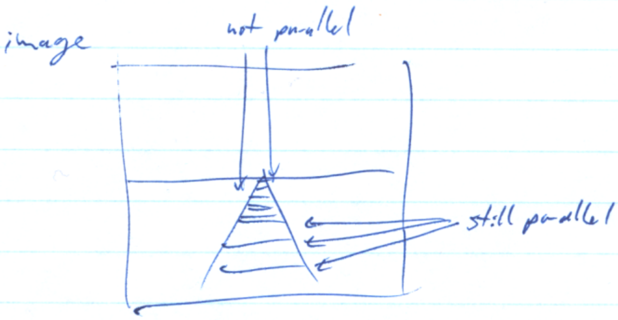
If $b_z = 0$, this is undefined. That corresponds to lines with constant z coord. Examining that case;

$$\begin{bmatrix} \frac{a_x + \lambda b_x}{a_z + \lambda b_z} \\ \frac{a_y + \lambda b_y}{a_z + \lambda b_z} \end{bmatrix} = \begin{bmatrix} \frac{a_x + \lambda b_x}{a_z} \\ \frac{a_y + \lambda b_y}{a_z} \end{bmatrix} = \frac{1}{a_z} \begin{bmatrix} a_x \\ a_y \end{bmatrix} + \lambda \frac{1}{a_z} \begin{bmatrix} b_x \\ b_y \end{bmatrix}$$

~~So such lines will remain parallel as their direction vectors~~

So such lines will remain parallel as their direction vectors are still proportional





| explanation
| diagram
| between diagram and explanation, depending on how well the section was done.

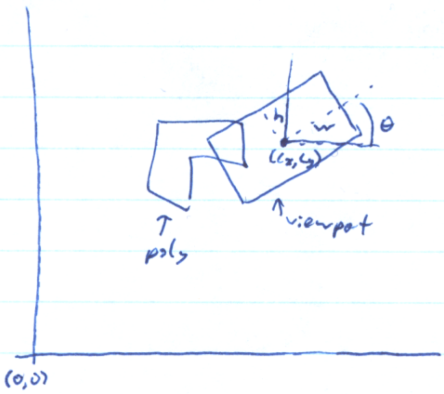
words explanation:

objects further away appear smaller, so the distance between parallel lines will appear to shrink as the points get further into the distance, thus no longer being parallel.

Conversely
→ if they do not go into the distance, they remain parallel

(followed by appropriate picture)

3



goal: transfm from world to viewport space, clip, transfm back.

assuming the operation operate on polygons;

$P = translate(-c_x, -c_y, P)$
$P = rotate(-\theta, P)$
$P = clip(-w, w, h, -h, P)$
$P = rotate(\theta, P)$
$P = translate(c_x, c_y, P)$

2 idea of untransform, clip, transform
1 retransform part
3 function details

$-\frac{1}{2}$ for $\theta, -\theta$ flip round ; rotates usually go ccw with the angle given
-1 for translate not present
-1 for rotate/translate order inversion
-0 for $\frac{h}{2}, \frac{w}{2}$ instead of h, w as if h->height, w->width, it should be halved (so original labeling ambiguous)