1. Transformations [10 marks]

A 2D affine transformation can be represented by:

$$X' = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} X$$

so we need to find a, b, c, d, e, f that satisfies:

$$\begin{bmatrix} 5 & 4 & 1 \\ 4 & -2 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 3 & 2 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

Solving the system gives us:

$a = 0$, $b = 1$, $c = 2$, $d = 4$, $e = 2$, $f = -10$

so the transformation is

$$\begin{bmatrix} 0 & 1 & 2 \\ 4 & 2 & -10 \\ 0 & 0 & 1 \end{bmatrix}$$

2. Animation [10 marks]

Assuming the pendulum is at $-k$ radians at $t = 0$

0 radians at $t = m/2$

k radians at $t = m$

Its rotation angle θ can be represented by:

$$\theta = k \cdot \sin\left(\frac{\pi}{m} t - \frac{\pi}{2}\right)$$

So the transformation (rotation along Z axis) is

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\left[k \cdot \sin\left(\frac{\pi}{m} t - \frac{\pi}{2}\right)\right] & -\sin\left[k \cdot \sin\left(\frac{\pi}{m} t - \frac{\pi}{2}\right)\right] & 0 & 0 \\ \sin\left[k \cdot \sin\left(\frac{\pi}{m} t - \frac{\pi}{2}\right)\right] & \cos\left[k \cdot \sin\left(\frac{\pi}{m} t - \frac{\pi}{2}\right)\right] & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(Other answers choosing different definition of θ is acceptable)

3. Viewing and Projection [20 marks]

a. The lens in a real camera reflects incoming
[5 marks]
rays and converges them to the image plane. Since an ideal pinhole camera does not exist in practice, a lens is needed to produce a sharp image with a finite opening.

The focal length affects the image size relative to the real object ($\nearrow$) and therefore the angle of view ($\searrow$). It also affects the depth of field (DOF) ($\searrow$).

The aperture affects the image brightness ($\nearrow$) and also the depth of field (DOF) ($\searrow$)

($\nearrow$ means positively correlated;
$\searrow$ means negatively correlated.)

b. Using the simple version of perspective projection:
[10 marks],

$$X' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/d & 0 \end{bmatrix} X \quad \text{we will get}$$

$$p' = \begin{bmatrix} p_x \\ p_y \\ p_z \\ p_z/d \end{bmatrix} = \begin{bmatrix} d\frac{p_x}{p_z} \\ d\frac{p_y}{p_z} \\ d \\ 1 \end{bmatrix} \quad q' = \begin{bmatrix} d\frac{q_x}{q_z} \\ d\frac{q_y}{q_z} \\ d \\ 1 \end{bmatrix} \quad m' = \begin{bmatrix} d\frac{(p_x+q_x)/2}{(p_z+q_z)/2} \\ d\frac{(p_y+q_y)/2}{(p_z+q_z)/2} \\ d \\ 1 \end{bmatrix}$$

in order for $m' = \frac{1}{2}(p'+q')$

we need
$$\begin{cases} d\left(\frac{p_x}{p_z} + \frac{q_x}{q_z}\right)/2 = d\frac{(p_x+q_x)}{(p_z+q_z)} \\ d\left(\frac{p_y}{p_z} + \frac{q_y}{q_z}\right)/2 = d\frac{(p_y+q_y)}{(p_z+q_z)} \\ d = d \end{cases}$$

Solving this we have

1° $p\_z = q\_z$ i.e. line segment $\overline{pq}$ is parallel to image plane

or 2° $p\_x / p\_z = q\_x / q\_z$ i.e. p and q are collinear with camera optical center.

and $p\_y / p\_z = q\_y / q\_z$

Otherwise $m' = \frac{1}{2}(p' + q')$ does not hold.

(If you use the complex form of projection matrix (with near & far), you won't get case 2°, that's fine)

4. Change of Basis [10 marks]

To change a point from camera 1 coordinates to world coordinates

$$P_{world} = A_1 P_1 \quad \text{where} \quad A_1 = \begin{bmatrix} u_{1x} & v_{1x} & w_{1x} & e_{1x} \\ u_{1y} & v_{1y} & w_{1y} & e_{1y} \\ u_{1z} & v_{1z} & w_{1z} & e_{1z} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

similarly $P_{world} = A_2 P_2$ where

$$A_2 = \begin{bmatrix} u_{2x} & v_{2x} & w_{2x} & e_{2x} \\ u_{2y} & v_{2y} & w_{2y} & e_{2y} \\ u_{2z} & v_{2z} & w_{2z} & e_{2y} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

therefore $P_2 = A_2^{-1} P_{world}$

$$= A_2^{-1} A_1 P_{world}$$