Today’s Topics

3. Transformations in 2D
4. Coordinate-free geometry
5. 3D Objects (curves & surfaces)
6. Transformations in 3D

Transformations

Transformation/Deformation in Graphics:
A function $f$, mapping points to points. simple transformations are usually invertible.

\[
\begin{bmatrix} x' & y' \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} T
\]

Applications:
• Placing objects in a scene.
• Composing an object from parts.
• Animating objects.

Processing Tree Demo!
https://processing.org/examples/tree.html

Representing 2D transforms as a 2x2 matrix

Rotate a point $[x 
 y] T$ by an angle $t$:

\[
\begin{bmatrix} x' & y' \end{bmatrix} = \begin{bmatrix} \cos t & -\sin t & 0 \\
\sin t & \cos t & 0 \end{bmatrix} \begin{bmatrix} x \\
y \end{bmatrix}
\]

Scale a point $[x 
 y] T$ by a factor $[s_x 
 s_y] T$:

\[
\begin{bmatrix} x' & y' \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\
0 & s_y & 0 \end{bmatrix} \begin{bmatrix} x \\
y \end{bmatrix}
\]

Translate:

Points as Homogeneous 2D Point Coords

\[
p = \begin{bmatrix} x \\
y \end{bmatrix} T = \begin{bmatrix} x \\
y \end{bmatrix} + \begin{bmatrix} 0 \\
0 \end{bmatrix} T
\]

\[
\begin{bmatrix} x' \\
y' \end{bmatrix} = \begin{bmatrix} x & y & 1 \end{bmatrix} T
\]

Topic 3:

2D Transformations

• Simple Transformations
• Homogeneous coordinates
• Homogeneous 2D transformations
• Affine transformations & restrictions

Let's start out simple...

Translate a point $[x 
 y] T$ by $[t_x 
 t_y] T$:

\[
x' = x + t_x \\
y' = y + t_y
\]

Rotate a point $[x 
 y] T$ by an angle $t$:

\[
x' = x \cos t - y \sin t \\
y' = x \sin t + y \cos t
\]

Scale a point $[x 
 y] T$ by a factor $[s_x 
 s_y] T$:

\[
x' = s_x x \\
y' = s_y y
\]
Cartesian \(\Leftrightarrow\) Homogeneous 2D Points

Cartesian \([x\ y]^T\) \(\Leftrightarrow\) Homogeneous \([x\ y\ 1]^T\)

Homogeneous \([x\ y\ w]^T\) \(\Rightarrow\) Cartesian \([x/w\ y/w\ 1]^T\)

Homogeneous points are equal if they represent the same Cartesian point. For eg: \([4\ -6\ 2]^T\ = [6\ -9\ -3]^T\).

What about \(w=0\)?

Points at \(\infty\) in Homogeneous Coordinates

\([x\ y\ w]^T\) with \(w=0\) represent points at infinity, though with direction \([x\ y]^T\) and thus provide a natural representation for vectors, distinct from points in Homogeneous coordinates.

Line Equations in Homogeneous Coordinates

A line given by the equation \(ax+by+c=0\)

can be represented in Homogeneous coordinates as:

\[l=[a\ b\ c]^T \Rightarrow l.p=[a\ b\ c]^T [x\ y\ 1]^T=0.\]

Aside: cross product as a matrix

\[
\begin{bmatrix}
0 & -c & b \\
c & 0 & -a \\
-a & b & 0
\end{bmatrix}
\]

The Line Passing Through 2 Points

For a line \(l\) that passes through two points \(p_0, p_1\)

we have \(l.p_0 = l.p_1 = 0\).

In other words we can write \(l\) using a cross product as:

\[l= p_0 \times p_1\]

Point of intersection of 2 lines

For a point that is the intersection of two lines \(l_0, l_1\)

we have \(p.l_0 = p.l_1 = 0\).

In other words we can write \(p\) using a cross product as:

\[p= l_0 \times l_1\]

Representing 2D transforms as a 3x3 matrix

Translate a point \([x\ y]^T\) by \([t_x, t_y]^T\):

\[
\begin{bmatrix}
x' \\
y' \\
1
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & t_x \\
0 & 1 & t_y \\
0 & 0 & 1
\end{bmatrix}
\]

Rotate a point \([x\ y]^T\) by an angle \(t\):

\[
\begin{bmatrix}
x' \\
y'
\end{bmatrix} =
\begin{bmatrix}
cost & -sint \\
sint & cost
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
\]

Scale a point \([x\ y]^T\) by a factor \([s_x, s_y]^T\):

\[
\begin{bmatrix}
x' \\
y'
\end{bmatrix} =
\begin{bmatrix}
s_x & 0 & 0 \\
0 & s_y & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
\]

What happens when the lines are parallel?
Properties of 2D transforms

...these 3x3 transforms have a variety of properties. Most generally they map lines to lines. Such invertible Linear transforms are also called Homographies.

...a more restricted set of transformations also preserve parallelism in lines. These are called Affine transforms.

...transforms that further preserve the angle between lines are called Conformal.

...transforms that additionally preserve the lengths of line segments are called Rigid.

Where do translate, rotate and scale fit into these?

Homography: mapping four points

How does the mapping of 4 points uniquely define the 3x3 Homography matrix?

Affine: preserving parallel lines

What restriction does the Affine property impose on H?

If two lines are parallel their intersection point at infinity, is of the form \([x \ y \ 0]^T\).

If these lines map to lines that are still parallel, then \([x \ y \ 0]^T\) transformed must continue to map to a point at infinity or \([x' \ y' \ 0]^T\)

i.e. \[
\begin{bmatrix}
  x' & y' & 0
\end{bmatrix}^T = \begin{bmatrix} A & t \end{bmatrix} \begin{bmatrix} x & y & 0 \end{bmatrix}^T
\]

In Cartesian co-ordinates Affine transforms can be written as:

\[
p' = Ap + t
\]
Affine properties: composition

Affine transforms are closed under composition. i.e. Applying transform \((A_1, t_1)(A_2, t_2)\) in sequence results in an overall Affine transform.

\[ p' = A_2(A_1p + t_1) + t_2 \Rightarrow (A_2A_1)p + (A_2t_1 + t_2) \]

Affine properties: inverse

The inverse of an Affine transform is Affine. - Prove it!

Affine transform: geometric interpretation

A change of basis vectors and translation of the origin

Point \(p\) in the local coordinates of a reference frame defined by \(<a_1, a_2, t>\) is

\[
\begin{bmatrix} a_1 & a_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} p \end{bmatrix}
\]

Affine transform: change of reference frame

How can we transform a point \(p\) from one reference frame \(<a_1, b_1, o_1>\), to another frame \(<a_2, b_2, o_2>\)?

Composing Transformations

Any sequence of linear transforms can be collapsed into a single 3x3 matrix by concatenating the transforms in the sequence.

In general transforms DO NOT commute, however certain combinations of transformations are commutative...

try out various combinations of translate, rotate, scale.

Rotation about a fixed point

The typical rotation matrix, rotates points about the origin. To rotate about specific point \(q\), use the ability to compose transforms...

\[ T_q R T_{-q} \]
Topic 4: Coordinate-Free Geometry (CFG)

- A brief introduction & basic ideas

CFG: dimension free geometric reasoning

- Points p [... 1]
- Vectors v [... 0]
- Lines l [....]

Dot products, Cross products,
Length of vectors,
Weighted average of points...

How do you find the angle between 2 vectors?

Topic 5: 3D Objects

3D parametric curves

\[ p(t) = (f_x(t), f_y(t), f_z(t)) \]

3D parametric surfaces

\[ p(t,s) = (f_x(t,s), f_y(t,s), f_z(t,s)) \]

3D parametric plane

\[ p(s,t) = q + as + tb \]
Topic 5:

3D Objects

- General curves & surfaces in 3D
- Normal vectors, surface curves & tangent planes
- Implicit surface representations
- Example surfaces: surfaces of revolution, bilinear patches, quadrics

Tangent / Normal vectors of 2D curves

Explicit: \( y = f(x) \)
Tangent is \( \frac{dy}{dx} \).

Parametric: \( x = f_x(t) \)
\( y = f_y(t) \)
Tangent is \( \left( \frac{dx}{dt}, \frac{dy}{dt} \right) \).

Implicit: \( f(x,y) = 0 \)
Normal is \( \nabla f \).

Direction of max. change

Given a tangent or normal vector in 2D how do we compute the other?

What about in 3D?

Normal vector of a plane

\( p(s,t) = q + as + tb \)

Normal vector of a plane

\( n = a \times b \)

Normal vector of a parametric surface

\( n = f'(u,v) \times f'(u,v) \)

Normal vector of a parametric surface
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  surfaces of revolution, bilinear patches, quadrics

Implicit function of a plane

\[ f(p) = (p-q).n = 0 \]

Implicit function: level sets

3D parametric surfaces

• Extrude
• Revolve
• Loft
• Square

Maya Live Demo...

3D parametric surfaces: Coons interpolation