Today’s Topics
3. Transformations in 2D
4. Coordinate-free geometry
5. 3D Objects (curves & surfaces)
6. Transformations in 3D

Topic 3:
2D Transformations

- Simple Transformations
- Homogeneous coordinates
- Homogeneous 2D transformations
- Affine transformations & restrictions

Transformations

Transformation/Deformation in Graphics:
A function \( f \), mapping points to points.
Simple transformations are usually invertible.

\[
\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & t_x \\ c & d & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}
\]

Applications:
- Placing objects in a scene.
- Composing an object from parts.
- Animating objects.

Processing Tree Demo!
https://processing.org/examples/tree.html

Let’s start out simple...

Translate a point \( [x, y]^T \) by \( [t_x, t_y]^T \):
\[
\begin{align*}
x' &= x + t_x \\
y' &= y + t_y
\end{align*}
\]

Rotate a point \( [x, y]^T \) by an angle \( t \):
\[
\begin{align*}
x' &= x \cos t - y \sin t \\
y' &= x \sin t + y \cos t
\end{align*}
\]

Scale a point \( [x, y]^T \) by a factor \( [s_x, s_y]^T \):
\[
\begin{align*}
x' &= s_x x \\
y' &= s_y y
\end{align*}
\]

Representing 2D transforms as a 2x2 matrix

Rotate a point \( [x, y]^T \) by an angle \( t \):
\[
\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}
\]

Scale a point \( [x, y]^T \) by a factor \( [s_x, s_y]^T \):
\[
\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & s_x \\ 0 & 0 & s_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}
\]

Translate?

Points as Homogeneous 2D Point Coords
### Cartesian ↔ Homogeneous 2D Points

Cartesian $[x \ y]^T$ ↔ Homogeneous $[x \ y \ 1]^T$

Homogeneous $[x \ y \ w]^T$ ↔ Cartesian $[x/w \ y/w \ 1]^T$

Homogeneous points are equal if they represent the same Cartesian point. For eg. $[4 \ -6 \ 2]^T = [6 \ -9 \ -3]^T$.

**What about $w=0$?**

### Points at $\infty$ in Homogeneous Coordinates

$[x \ y \ w]^T$ with $w=0$ represent points at infinity, though with direction $[x \ y]^T$ and thus provide a natural representation for vectors, distinct from points in Homogeneous coordinates.

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### Line Equations in Homogeneous Coordinates

A line given by the equation $ax + by + c = 0$ can be represented in Homogeneous coordinates as:

$\mathbf{l} = [a \ b \ c]^T$, making the line equation

$\mathbf{l} \cdot \mathbf{p} = 0$.

**Aside:** cross product as a matrix

$\begin{bmatrix} 0 & -c & b \\ b & 0 & -a \\ -c & a & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0$

### The Line Passing Through 2 Points

For a line $\mathbf{l}$ that passes through two points $\mathbf{p}_0, \mathbf{p}_1$

we have $\mathbf{l} \cdot \mathbf{p}_0 = \mathbf{l} \cdot \mathbf{p}_1 = 0$.

In other words we can write $\mathbf{l}$ using a cross product as:

$\mathbf{l} = \mathbf{p}_0 \times \mathbf{p}_1$

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### Point of intersection of 2 lines

For a point that is the intersection of two lines $\mathbf{l}_0, \mathbf{l}_1$

we have $\mathbf{p} \cdot \mathbf{l}_0 = \mathbf{p} \cdot \mathbf{l}_1 = 0$.

In other words we can write $\mathbf{p}$ using a cross product as:

$\mathbf{p} = \mathbf{l}_0 \times \mathbf{l}_1$

What happens when the lines are parallel?

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### Representing 2D transforms as a 3x3 matrix

**Translate a point $[x \ y]^T$ by $[t_x \ t_y]^T$**:

$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$

**Rotate a point $[x \ y]^T$ by an angle $t$**:

$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$

**Scale a point $[x \ y]^T$ by a factor $[s_x \ s_y]^T$**:

$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$

---
Properties of 2D transforms

...these 3x3 transforms have a variety of properties. Most generally, they map lines to lines. Such invertible linear transforms are also called **Homographies**.

...a more restricted set of transformations also preserve parallelism in lines. These are called **Affine** transforms.

...transforms that further preserve the angle between lines are called **Conformal**.

...transforms that additionally preserve the lengths of line segments are called **Rigid**.

Where do translate, rotate and scale fit into these?

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Homography: mapping four points

How does the mapping of 4 points uniquely define the 3x3 Homography matrix?

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Affine: preserving parallel lines

What restriction does the Affine property impose on H?

If two lines are parallel, their intersection point at infinity, is of the form \([x \ y \ 0]^T\).

If these lines map to lines that are still parallel, then \([x \ y \ 0]^T\) transformed must continue to map to a point at infinity or \([x' \ y' \ 0]^T\)

i.e. \([x' \ y' \ 0]^T = \begin{bmatrix} A & b \end{bmatrix} [x \ y \ 0]^T\)

In Cartesian co-ordinates Affine transforms can be written as:

\[ p' = Ap + t \]
Affine properties: composition

Affine transforms are closed under composition. i.e.
Applying transform \((A_1,t_1)(A_2,t_2)\) in sequence results in an overall
Affine transform.

\[
p' = A_2(A_1p + t_1) + t_2 \Rightarrow (A_2A_1)p + (A_2t_1 + t_2)
\]

Affine properties: inverse

The inverse of an Affine transform is Affine.
- Prove it!

Affine transform: geometric interpretation

A change of basis vectors and translation of the origin

![Diagram](A change of basis vectors and translation of the origin)

Point \(p\) in the local coordinates of a reference frame defined by \(<a_1,a_2,t>\) is

\[
\begin{pmatrix}
a_1 & a_2 \\
0 & 0 \\
1 & 1
\end{pmatrix}
\begin{pmatrix}
p \\
t
\end{pmatrix}
\]

Affine transform: change of reference frame

How can we transform a point \(p\) from one reference frame \(<a_1,b_1,o_1>\),
to another frame \(<a_2,b_2,o_2>\)?

![Diagram](How can we transform a point \(p\) from one reference frame \(<a_1,b_1,o_1>\),
to another frame \(<a_2,b_2,o_2>\)?)

Composing Transformations

Any sequence of linear transforms can be collapsed into a single
3x3 matrix by concatenating the transforms in the sequence.

In general transforms DO NOT commute, however certain combinations
of transformations are commutative...

try out various combinations of translate, rotate, scale.

Rotation about a fixed point

The typical rotation matrix, rotates points about the origin.
To rotate about specific point \(q\), use the ability to compose transforms...

\[
T_qRT_q
\]
Topic 4:

Coordinate-Free Geometry (CFG)

- A brief introduction & basic ideas

CFG: dimension free geometric reasoning

- Points \( p \) [... 1]
- Vectors \( v \) [... 0]
- Lines \( l \) [……]

- Dot products, Cross products,
- Length of vectors,
- Weighted average of points...

How do you find the angle between 2 vectors?

Topic 5:

3D Objects

- General curves & surfaces in 3D
- Normal vectors, surface curves & tangent planes
- Implicit surface representations
- Example surfaces:
  - surfaces of revolution, bilinear patches, quadrics

3D parametric curves

\[ p(t) = (f_x(t), f_y(t), f_z(t)) \]

3D parametric surfaces

\[ p(s,t) = q + a \cdot s + b \cdot t \]

3D parametric plane
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Tangent / Normal vectors of 2D curves

Explicit: \( y = f(x) \)
Parameteric: \( x = f_1(t) \)
\( y = f_2(t) \)
Implicit: \( f(x,y) = 0 \)

Tangent is \( \frac{dy}{dx} \).
Tangent is \( (\frac{dx}{dt}, \frac{dy}{dt}) \).
Normal is \( \text{gradient}(f) \).
direction of max. change

Given a tangent or normal vector in 2D how do we compute the other?

What about in 3D?

Normal vector of a plane

\[ p(s,t) = q + as + tb \]

Normal vector of a plane

\( n = a \times b \)

Normal vector of a parametric surface

\[ n = \frac{\partial f}{\partial u} \times \frac{\partial f}{\partial v} \]

Normal vector of a parametric surface

\[ n = f'(u,v) \times f'(u,v) \]
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Implicit function of a plane

\[ f(p) = (p-q) \cdot n = 0 \]

Implicit function: level sets

3D parametric surfaces

- Extrude
- Revolve
- Loft
- Square

Maya Live Demo...

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3D parametric surfaces: Coons interpolation