Topic:

Catmull-Romm Splines
An n-degree polynomial in t has n+1 coefficients. It can be defined by n+1 constraints. A line (degree 1) thus needs two points.

\[ p(t) = (1-t)p_0 + tp_1 \]

\[ p(t) = [t \ 1] \begin{bmatrix} -1 & 1 \end{bmatrix} [p_0 \ p_1]^T \]

\[ \begin{bmatrix} 1 & 0 \end{bmatrix} \]
For smooth (tangent continuous) interpolation across points, we need to be able to interpolate points as well as tangents.

Two points $p_0$, $p_1$, and two tangents $p'_0$, $p'_1$, define a cubic (degree 3) curve.
Designing Polynomial Curves from constraints

\( p(t) = TA \), where \( T \) is powers of \( t \). for a cubic \( T = [t^3 \ t^2 \ t^1 \ 1] \).

Written with geometric constraints \( p(t) = TMG \), where \( M \) is the **Basis matrix** of a design curve and \( G \) the specific design constraints.

An example of constraints for a cubic Hermite for eg. are end points and end tangents. i.e. \( p_0, p'_0 \) at \( t=0 \) and \( p_1, p'_1 \) at \( t=1 \). Plugging these constraints into \( p(t) = TA \) we get.

\[
\begin{align*}
\mathbf{B} & \quad p(0) = p_0 = [\ 0 \ 0 \ 0 \ 1 \] A_h \\
p(1) = p_1 = [\ 1 \ 1 \ 1 \ 1 \] A_h \\
p'(0) = p'_0 = [\ 0 \ 0 \ 1 \ 0 \] A_h & \Rightarrow \quad G=BA, \ A=MG \Rightarrow M=B^{-1} \\
p'(1) = p'_1 = [\ 3 \ 2 \ 1 \ 0 \] A_h
\end{align*}
\]
Hermite Basis Matrix

\[
\begin{bmatrix}
0 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 \\
3 & 2 & 1 & 0
\end{bmatrix}
= \begin{bmatrix}
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 \\
-2 & 2 & 1 & 1 \\
-3 & 3 & -2 & -1 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix}
\]

The columns of the Basis Matrix form Basis Functions such that:

\[
p(t) = (2t^3 -3t^2 +1)p_0 + (-2t^2 +3t^2)p_1 + (t^3 -2t^2 +t)p'_0 + (t^3 -t^2)p'_1.
\]
Interpolation: Catmull-Romm Splines

Catmull-Romm Interpolation

Pick tangents based on a factor $k$ (1/2 for eg.) of the vector between neighbor points.

$p'_i = k*(p_{i+1} - p_{i-1})$.

For the end-points there is only one neighbor:

$p'_0 = k*(p_1 - p_0)$.
$p'_n = k*(p_n - p_{n-1})$. 