Q1.

\[ x(t) = at \]
\[ y(t) = -\frac{1}{2}gt^2 + bt + h \]

a) Tangent vector is \( f'(t) = (x'(t), y'(t)) \)
\[ x'(t) = a \]
\[ y'(t) = -gt + b \]

Normal vector: \((-y'(t), x(t)) = (gt-b, a)\)

b) The impact occur when \( y(t) = 0 \) \( \iff \) \(-\frac{1}{2}gt^2 + bt + h = 0\)
so \[ t = \frac{-b \pm \sqrt{b^2 + 2gh}}{-g} \]
\[ = \frac{b \pm \sqrt{b^2 + 2gh}}{g} \]

Since \( h > 0, g > 0 \), we have \( \sqrt{b^2 + 2gh} > b \), since \( t > 0 \)
\[ t = \frac{b + \sqrt{b^2 + 2gh}}{g} \]

Location: \[ x(t) = at = a \left( \frac{b + \sqrt{b^2 + 2gh}}{g} \right) = \frac{ab + a\sqrt{b^2 + 2gh}}{g} \]
\[ y(t) = 0 \]

Velocity: \( x'(t) = a \)
\[ y'(t) = -g \left( \frac{b + \sqrt{b^2 + 2gh}}{g} \right) + b \]
\[ = -\sqrt{b^2 + 2gh} \]
Q2.

(a) Translation: \( T = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \)

Uniform scale: \( S = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} \)

\( T \cdot S = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} = \begin{bmatrix} s & 0 & t_x \\ 0 & s & t_y \\ 0 & 0 & s \end{bmatrix} \)

\( S \cdot T = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} s & 0 & st_x \\ 0 & s & st_y \\ 0 & 0 & s \end{bmatrix} \)

\( T \cdot S \neq S \cdot T \)

\(:= \text{Not commute} \)

(b) Two different rotations,

\( A = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \)

\( A \cdot B - B \cdot A = \begin{bmatrix} \cos \theta \cos \phi & -\sin \theta \cos \phi & \sin \phi \\ \sin \theta \cos \phi & \cos \theta \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi & 1 \end{bmatrix} \begin{bmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos \theta \cos \phi & -\sin \theta \cos \phi - \sin \phi \cos \phi & \sin \phi \\ \sin \theta \cos \phi & \cos \theta \cos \phi + \sin \theta \sin \phi & -\sin \phi \\ \sin \phi & \cos \phi + \sin \phi \cos \theta & 1 \end{bmatrix} \)

\( B \cdot A = \begin{bmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos \theta \cos \phi & -\sin \theta \cos \phi - \sin \phi \cos \phi & \sin \phi \\ \sin \theta \cos \phi & \cos \theta \cos \phi + \sin \theta \sin \phi & -\sin \phi \\ \sin \phi & \cos \phi + \sin \phi \cos \theta & 1 \end{bmatrix} = A \cdot B \)

\( : = \text{Commute.} \)
c) \( X \) shear and uniform scaling
\[
x\text{-shear: } X_S = \begin{bmatrix} 1 & m & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad S = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]
\[
X_S \cdot S = \begin{bmatrix} 1 & m & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} s & sm & 0 \\ 0 & s & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]
\[
S \cdot X_S = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & m & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} s & sm & 0 \\ 0 & s & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]
\[
X_S \cdot S = S \cdot X_S
\]
\[\boxed{\text{Commutative}}\]

d) \( X \) shear and non-uniform scaling
\[
x\text{-shear: } X_S = \begin{bmatrix} 1 & m & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad S = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]
\[
X_S \cdot S = \begin{bmatrix} 1 & m & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} s_x & s_xm & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]
\[
S \cdot X_S = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & m & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} s_x & s_xm & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]
\[
\text{Since } s_x \neq s_y, \quad X_S \cdot S \neq S \cdot X_S
\]
\[\boxed{\text{Not Commute}}\]
Q3
One possible sequence of transformation is:
Reflect about y axis
\[ T_1 = \begin{bmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} \]
Uniform scale by 2
\[ T_2 = \begin{bmatrix}
2 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 1
\end{bmatrix} \]
Rotate by \(-\pi/2\)
\[ T_3 = \begin{bmatrix}
\cos(-90) & -\sin(-90) & 0 \\
\sin(-90) & \cos(-90) & 0 \\
0 & 0 & 1
\end{bmatrix} = \begin{bmatrix}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix} \]
Translate up by 6 units
\[ T_4 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 6 \\
0 & 0 & 1
\end{bmatrix} \]

The transformation can be done with one single transformation
\[ T = T_4 \times T_3 \times T_2 \times T_1 \]
\[ = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 6 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
2 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} \]
\[ = \begin{bmatrix}
0 & 2 & 0 \\
2 & 0 & 6 \\
0 & 0 & 1
\end{bmatrix} \]
Q4

a) Let \( H = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & 1 \end{bmatrix} \) be the homographic matrix that maps the corresponding points.

For any point \((x, y)\) that is mapped to point \((s, t)\), we will have the following transformation:

\[
\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} s \\ t \\ 1 \end{bmatrix}
\]

Thus

\[
ax + by + c = s(gx + hy + 1) \\
dx + ey + f = t(gx + hy + 1)
\]

Apply the mapping from the question:

\((0, 0), (1, 0), (0, 1), (1, 1)\) to points \((-4, 2), (-3, 0), (1, -7), (0, -5)\), respectively.

We will have a set of equations

\[
\begin{align*}
c &= -4 \\
f &= 2 \\
a &= -3g + 1 \\
d &= -2 \\
b &= h + 5 \\
e &= -7h - 9 \\
-3g + h &= -2 \\
5g - 2h &= 4
\end{align*}
\]

After solving the system, we will have the matrix

\[
H = \begin{bmatrix} 1 & 3 & -4 \\ -2 & 5 & 2 \\ 0 & -2 & 1 \end{bmatrix}
\]

b) Apply point \((1, 2)\) to the transformation:

\[
\begin{bmatrix} 1 & 3 & -4 \\ -2 & 5 & 2 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 10 \\ -3 \end{bmatrix} = \begin{bmatrix} -1 \\ \frac{-10}{3} \\ 1 \end{bmatrix}
\]

The point is \((-1, -(10/3))\)
c) This is not affine transformation.

Reason:

The transformation does not preserve parallels. For example, line through point (0, 0) and (1, 0) and the line through point (0, 1) and (1, 1) are parallel, while the points through (-4, 2), (-3, 0) and (1, -7), (0, -5) are not parallel.

This answer is also accepted:

The last row of the matrix is not (0, 0, 1).
Q5.

Translation: \[
\begin{bmatrix}
1 & 0 & tx \\
0 & 1 & ty \\
0 & 0 & 1
\end{bmatrix} = T
\]

Rotation: \[
\begin{bmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{bmatrix} = R
\]

Scaling: \[
\begin{bmatrix}
S_x & 0 & 0 \\
0 & S_y & 0 \\
0 & 0 & 1
\end{bmatrix} = S
\]

\[
S \cdot R \cdot T = \begin{bmatrix}
-3 & 2 & 6 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

\[
\begin{bmatrix}
S_x & 0 & 0 \\
0 & S_y & 0 \\
0 & 0 & 1
\end{bmatrix} \cdot \begin{bmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{bmatrix} \cdot \begin{bmatrix}
1 & 0 & tx \\
0 & 1 & ty \\
0 & 0 & 1
\end{bmatrix} = \begin{bmatrix}
-3 & 2 & 6 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

\[
\begin{bmatrix}
S_x \cos \theta & -S_x \sin \theta & S_x (\cos \theta \cdot \cos \theta - \sin \theta \cdot \sin \theta) \\
S_y \sin \theta & S_y \cos \theta & S_y (\cos \theta \cdot \sin \theta + \sin \theta \cdot \cos \theta) \\
0 & 0 & 1
\end{bmatrix} = \begin{bmatrix}
0 & 2 & 6 \\
-3 & 0 & -6 \\
0 & 0 & 1
\end{bmatrix}
\]

1. \[S_x \cdot \cos \theta = 0 \quad \Rightarrow \quad \text{Not possible to have a scaling.} \quad \therefore \cos \theta = 0, \ \theta = \frac{\pi}{2}, \frac{3\pi}{2}\]
2. \[-S_x \cdot \sin \theta = 2\]
3. \[S_x (\cos \theta \cdot \cos \theta - \sin \theta \cdot \sin \theta) = 6\]
4. \[S_y \cdot \sin \theta = -3\]
5. \[S_y \cdot \cos \theta = 0\]
6. \[S_y (\cos \theta \cdot \sin \theta + \sin \theta \cdot \cos \theta) = -6\]
case 1: \( \Theta = \frac{\pi}{2} \)

- \(-S_x \cdot \sin \left( \frac{\pi}{2} \right) = 2\)
  \(-S_x = 2\)
  \(S_x = -2\)

- \(S_y \cdot \sin \left( \frac{\pi}{2} \right) = -3\)
  \(S_y = -3\)

- From (3) \(-2(t_x \cos \left( \frac{\pi}{2} \right) - t_y \sin \left( \frac{\pi}{2} \right)) = 6\) \(\Rightarrow tx = 2\)
- From (1) \(-3(t_x \sin \left( \frac{\pi}{2} \right) - t_y \cos \left( \frac{\pi}{2} \right)) = -6\) \(\Rightarrow ty = 3\)

: \(T = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \quad R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad S = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \)

case 2: \( \Theta = \frac{3\pi}{2} \)

Similarly as above, we can get:

\(T = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \quad R = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad S = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \)