CSC 236 November 22, 2004
St George Campus University of Toronto

Solutions for Assignment #4

**Answer to Question 1.**

**a.**

Statement: $\forall x\, \exists y\, \exists z\, \big(S(y,z,x) \land \neg{\approx}(y,z)\big)$

In English: Every number is the sum of two distinct numbers.

This statement is false in $\mathcal{N}$, because the only pair of natural numbers whose sum is 0, are 0 and 0, and these are not distinct. The statement is true in $\mathcal{Z}$, because every integer $x$ can be written as the sum of $x-1$ and 1, which are distinct if $x \neq 2$; and 2 can be written as the sum of the distinct integers 0 and 2.

**b.**

Statement: $\exists x\, \forall y\, \big(\neg{\approx}(y,x) \rightarrow L(y,x)\big)$

In English: There is a number bigger than all others.

This is obviously false in both $\mathcal{N}$ and $\mathcal{Z}$ because $x+1$ is bigger than any supposed maximum element $x$.

**c.**

Statement: $\forall x\, \forall y\, \Big(\big(L(x,0) \land P(x,x,y)\big) \rightarrow L(0,y)\Big)$

In English: The square of every negative number is positive.

This is true in $\mathcal{N}$ trivially (since there is no negative $x$ in $\mathbb{N}$) and also true in $\mathcal{Z}$ because the square of every negative integer is positive.

**d.**

$\forall x\, \forall y\, \forall z\, \forall u\, \forall v\, \Big(\big(L(x,y) \land \neg{\approx}(z,\boldsymbol{0}) \land P(x,z,u) \land P(y,z,v)\big) \rightarrow L(u,v)\Big)$

In English: For every pair of numbers the first of which is less than the second, the product of the first by a number other than zero is less than the product of the second by the same number.

Or: Given any two numbers where the first is less than the second, the product of the first with some non-zero number will be less than the product of the second with that same non-zero number. (this is derived from a student answer)

This is obviously true in $\mathcal{N}$ (multiplying both sides of an inequality by the same positive integer also yields an inequality in the same direction), but false in $\mathcal{Z}$ (e.g., we get a counterexample if $x=1$, $y=2$ and $z=-1$).

**e.**

$\exists x\, \forall y\, \big(L(y,x) \rightarrow \exists x\, S(x,\boldsymbol{1},x)\big)$

In English: There is a number such that for all smaller numbers, some number is equal to its successor.

This is true in $\mathcal{N}$ and false in $\mathcal{Z}$.

First consider $\mathcal{N}$: Since there is no natural number $y<0$, the formula $\forall y\, \big(L(y,x) \rightarrow \exists x\, S(x,\boldsymbol{1},x)\big)$ is trivially true for $x=0$, and so the formula $\exists x\, \forall y\, \big(L(y,x) \rightarrow \exists x\, S(x,\boldsymbol{1},x)\big)$ is true in $\mathcal{N}$. Next consider $\mathcal{Z}$: For every $x$, the formula $\forall y\, \big(L(y,x) \rightarrow \exists x\, S(x,\boldsymbol{1},x)\big)$ is false (because for $y=x-1$ the antecedent is true while the consequent is false). Therefore the formula, $\exists x\, \forall y\, \big(L(y,x) \rightarrow \exists x\, S(x,\boldsymbol{1},x)\big)$ is false in $\mathcal{Z}$.

**Answer to Question 2.**

**a.** This statement is true. We have

$$
\begin{array}{lll}
 & \neg\forall x\,\big(A(x) \to B(x)\big) & \\
\text{LEQV} & \exists x\,\neg\big(A(x) \to B(x)\big) & \text{[Duality]} \\
\text{LEQV} & \exists x\,\neg\big(\neg A(x) \vee B(x)\big) & [\to \text{ law}] \\
\text{LEQV} & \exists x\,\big(A(x) \wedge \neg B(x)\big) & \text{[DeMorgan and Double Negation]}
\end{array}
$$

**b.** This statement is false. Consider the structure whose domain is the set of odd numbers, and which interprets $A(x)$ as the predicate "$x$ is prime" and $B(x)$ as the predicate "$x$ is even". In this structure the first formula is false: $\exists x\, A(x)$ is true, but $\exists x\, B(x)$ is false. On the other hand, in this structure the second formula is true: If $x$ is any nonprime odd number (such as 9), the formula $A(x) \to B(x)$ is true, and therefore $\exists x\,\big(A(x) \to B(x)\big)$ is true.

**c.** This statement is true because both formulas are valid, i.e., true in all interpretations (given that the domain is required to be non-empty). The first formula states that either some $x$ satisfies $A$ or some $x$ falsifies $A$; this is true no matter what predicate $A$ stands for, because the domain of every structure is nonempty and any element of the domain must either satisfy or falsify $A$. The second formula states that every $x$ either satisfies or falsifies $A$, which again is true no matter what predicate $A$ stands for.

**d.** This statement is false. As just argued, the second formula is valid. The first, however is not. For example, consider the structure whose domain is $\mathbb{N}$ and suppose $A(x)$ stands for the predicate "$x$ is even". The first formula is false in this structure, because it is not true that all natural numbers are even nor is it true that all natural numbers are not even.

**e.** This statement is false. For a counterexample, let $E = A(x)$, $F = \neg\forall y\, A(y)$. Note that $x$ does not appear free in $F$. In this case, the first formula is $\forall x\, A(x) \oplus \neg\forall y\, A(y)$, which is obviously valid (true in all interpretations). The second formula is $\forall x\,\big(A(x) \oplus \neg\forall y\, A(y)\big)$, which is not valid. To see this, consider the structure whose domain is $\mathbb{N}$, and which interprets $A(x)$ as the predicate "$x$ is prime". If we take $x = 17$ (or any prime number), the formula $A(x) \oplus \neg\forall y\, A(y)$ is false (because $A(x)$ and $\neg\forall y\, A(y)$ are both true). Therefore, the formula $\forall x\,\big(A(x) \oplus \neg\forall y\, A(y)\big)$ is false in this structure and so it is not valid.

**Answer to Question 3.**

**a.** Statement (i) is ambiguous. One way of translating it is:

$$\forall x\,\Big(P(x) \to \exists y\,\big(T(y) \wedge F(x,y)\big)\Big)$$

which is logically equivalent to

$$\forall x \exists y\,\Big(P(x) \to \big(T(y) \wedge F(x,y)\big)\Big)$$

This says that for each person there is a time at which you can fool him/her. Note that in this interpretation, different people may be fooled at different times.

Another way of translating statement (i) is:

$$\exists y\,\Big(T(y) \wedge \forall x\,\big(P(x) \to F(x,y)\big)\Big)$$

which is logically equivalent to

$$\exists y\,\forall x\,\Big(T(y) \wedge \big(P(x) \to F(x,y)\big)\Big)$$

This way of translating the first statement is stronger than the other: it says that there is a time when you can fool all people. In this interpretation all people are fooled at the same time. One of us suspects that Lincoln intended the second, stronger interpretation, referring to an historical time at which "everyone" was gullible. Another of us has always understood the statement to mean the weaker interpretation, in which there is not necessarily any special time of universal deception. This illlustrates the value of precision!

Statement (ii) is also ambiguous. It can be translated as one of:

$$\exists x\, \forall y\, \Big(P(x) \wedge \big(T(y) \to F(x,y)\big)\Big) \qquad \text{or} \qquad \forall y\, \exists x\, \Big(P(x) \to \big(T(y) \wedge F(x,y)\big)\Big)$$

The first says that someone can be fooled all the time (the same person always). The second is weaker and says that at any time, someone can be fooled; now, however, it may be different people who can be fooled at different times. In this case, all of us suspect that Lincoln had the stronger meaning in mind, namely that there are some individuals who are always gullible.

Statement (iii) can be translated as:

$$\neg \forall x\, \forall y\, \Big(\big(P(x) \wedge T(y)\big) \to F(x,y)\Big)$$

An equivalent way of saying the same thing is

$$\exists x\, \exists y\, \big(P(x) \wedge T(y) \wedge \neg F(x,y)\big)$$

that is, there is a person and a time such that this person cannot be fooled at that time.

**b.** (i) $\forall x\, L(x, \boldsymbol{b}) \wedge \forall y\, \big(L(\boldsymbol{b}, y) \to {\approx}(y, \boldsymbol{i})\big)$.

(ii) Suppose that this conjunctive statement is true. Then, in particular, the first conjunct is true, namely $\forall x\, L(x, \boldsymbol{b})$. Since $L(x, \boldsymbol{b})$ is true for all individuals $x$, it is true, in particular, for individual $\boldsymbol{b}$ — i.e., the statement $L(\boldsymbol{b}, \boldsymbol{b})$ is true (my baby loves my baby). But also the second conjunct is true, namely $\forall y\, \big(L(\boldsymbol{b}, y) \to {\approx}(y, \boldsymbol{i})\big)$. Since this is true for all $y$, it is again true in particular for $y = \boldsymbol{b}$. In other words, it is true that $L(\boldsymbol{b}, \boldsymbol{b}) \to {\approx}(\boldsymbol{b}, \boldsymbol{i})$. But we know that the antecedent of this implication is true. Hence, the consequent ${\approx}(\boldsymbol{b}, \boldsymbol{i})$ must also be true. In other words, it is true that $\boldsymbol{b}$ ("my baby") is the same person as $\boldsymbol{i}$ ("me").

Assuming that the statement $(*)$ is true, we proved that my baby is me. In other words, $(*)$ logically implies that my baby is me.

(Of course, "everybody" in the song lyrics really meant "everyone other than $\boldsymbol{b}$" (or even a smaller set of people). In this case, unlike in part **a**, the imprecision of the English statement does not cause any confusion, only the opportunity for some fun.)

**Answer to Question 4.**
**Note:** Different orders of applying the factoring rules could result in different orders for the quantifiers.
**Note 2:** The factoring laws are equivalences and thus work both ways. You might think of their uses below as "backwards".

**a.**

$$\begin{array}{lll}
 & \forall x\, R(x) \to \big(\exists x\, P(x) \vee \exists x\, Q(x)\big) & \\
\text{LEQV} & \exists x\, \Big(R(x) \to \big(\exists x\, P(x) \vee \exists x\, Q(x)\big)\Big) & \text{[factoring]} \\
\text{LEQV} & \exists x\, \Big(R(x) \to \big(\exists u\, P(u) \vee \exists v\, Q(v)\big)\Big) & \text{[2 x renaming]} \\
\text{LEQV} & \exists x\, \Big(R(x) \to \exists u\, \exists v\, \big(P(u) \vee Q(v)\big)\Big) & \text{[2 x factoring]} \\
\text{LEQV} & \exists x\, \exists u\, \exists v\, \Big(R(x) \to \big(P(u) \vee Q(v)\big)\Big) & \text{[2 x factoring]}
\end{array}$$

**b.**

| | | |
|---|---|---|
| | $\neg\forall x\, S(x,y) \leftrightarrow \exists y\, T(x,y,z)$ | |
| LEQV | $\neg\forall u\, S(u,y) \leftrightarrow \exists v\, T(x,v,z)$ | [2 x renaming] |
| LEQV | $\exists u\, \neg S(u,y) \leftrightarrow \exists v\, T(x,v,z)$ | [duality] |
| LEQV | $\big(\exists u\, \neg S(u,y) \wedge \exists v\, T(x,v,z)\big) \vee \big(\neg\exists u\, \neg S(u,y) \wedge \neg\exists v\, T(x,v,z)\big)$ | [$\leftrightarrow$ law] |
| LEQV | $\big(\exists u\, \neg S(u,y) \wedge \exists v\, T(x,v,z)\big) \vee \big(\forall u\, S(u,y) \wedge \forall v\, \neg T(x,v,z)\big)$ | [2 x duality & double negation] |
| LEQV | $\big(\exists u\, \neg S(u,y) \wedge \exists v\, T(x,v,z)\big) \vee \big(\forall p\, S(p,y) \wedge \forall q\, \neg T(x,q,z)\big)$ | [2 x renaming] |
| LEQV | $\exists u\, \exists v\, \forall p\, \forall q\, \Big(\big(\neg S(u,y) \wedge T(x,v,z)\big) \vee \big(S(p,y) \wedge \neg T(x,q,z)\big)\Big)$ | [4 x factoring] |

**Answer to Question 5.**

**a.** $\exists c\, \Big(\mathit{Course}(c,t) \wedge \exists s\, \big(\exists a\, \mathit{Student}(s, \text{“Euclid”}, a) \wedge \exists y\, \exists m\, \mathit{Took}(s,c,y,m)\big)\Big)$

**b.** $\exists c\, \Bigg(\mathit{Course}(c,t) \wedge \exists s\, \Big(\exists a\, \mathit{Student}(s, \text{“Euclid”}, a) \wedge \exists y\, \big(\exists m\, \mathit{Took}(s,c,y,m) \wedge \mathit{Taught}(\text{“Pythagoras”}, c, y)\big)\Big)\Bigg)$

**c.** $\exists s\, \exists a\, \exists s' \Big(\mathit{Student}(s, \text{“Euclid”}, a) \wedge \mathit{Student}(s', n, a')$

$\qquad \wedge \exists c\, \exists y\, \exists m\, \exists m'\, \big(\mathit{Took}(s,c,y,m) \wedge \mathit{Took}(s',c,y,m') \wedge \neg\approx(m, \text{“A”}) \wedge \approx(m', \text{“A”})\big)\Big)$

**d.** $\mathit{Course}(c,t) \wedge \exists y\, \mathit{Taught}(\text{“Pythagoras”}, c, y) \wedge \forall y\, \forall p\, \big(\mathit{Taught}(p,c,y) \rightarrow \approx(p, \text{“Pythagoras”})\big)$

**e.** $\exists c\, \big(\mathit{Course}(c, \text{“Geometry”}) \wedge \mathit{Taught}(\text{“Pythagoras”}, c, y)\big)$
$\wedge \forall s\, \forall c\, \forall m\, \Big(\big(\mathit{Taught}(\text{“Pythagoras”}, c, y) \wedge \mathit{Took}(s,c,y,m)\big) \rightarrow \neg\approx(m, \text{“F”})\Big)$