CSC260S: Intro to Scientific, Graphical and Symbolic Computing
Department of Computer Science
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Instructor: E. Fiume **Term Test 1** **22 Feb. 1994**

Closed book. The test questions continue overleaf. Show your work. The test is out of 45.

1. [5] Reparameterise the polynomial

$$p(x) \; = \; x^2 - x + 1$$

from the interval $x \in [-1,2]$ to the identical polynomial $\hat{p}(t)$ on the interval $t \in [0,1]$.

2. [10] Let f be an arbitrary quadratic equation of the form $f(t) = at^2 + bt + c$. For any function f, we can approximate the value of f at time instant $t_{i+1} = t_i + h$, for a small constant time step h, if we know the value of f and its derivative at time t_i:

$$f(t_{i+1}) \approx f(t_i) + \Delta f\,(t_i).$$

$\Delta f(t)$ is the first forward difference of f.

(a) [6] Derive the necessary forward differences for a quadratic function with respect to variable t and step size h. By ‘‘necessary’’ we mean the forward differences $\Delta^1 f$, $\Delta^2 f$, $\Delta^3 f$, etc., as required until the final forward difference is a constant. Simplify your results by grouping together terms with respect to powers of t.

(b) [4] Outline an algorithm that computes the values $f(t_i)$ on the quadratic curve, where $t_i = ih = 0, h, 2h, \cdots, nh = 1$, for $i = 0, 1, \cdots, n$. Note that $h = \frac{1}{n}$.

3. [8] The optimal line rendering between (0,0) and (x,y), x,y integers and $y \le x$, is

$$\left\{ (i,j) : 0 \le i \le x, \;\; j = \left\langle \frac{y}{x} i \right\rangle \right\}.$$

The notation $\langle z \rangle$ denotes the *round* of z, defined as floor($z + ½$) for positive $z \in \mathbf{R}$, and floor($z - ½$) for negative z. We shall analyse the error behaviour of this rendering. Define ε_i to be the *vertical error* made in column i by the optimal line rendering, where $0 \le i \le x$. That is, for each such i, ε_i is the (signed) difference between the real y value of the line segment at pixel i and the integral value of y at (i,j) chosen by the rendering.

(a) [2] Write down a simple mathematical expression for ε_i.

(b) [6] (Not hard, but show your reasoning.) For a line segment from (0,0) to (x,y), show that

$$|\varepsilon_i| \; = \; |\varepsilon_{x-i}|.$$

4. [10] Suppose we wish to represent parametric piecewise cubic curve segments over [0,1]. Recall that a way of representing such curves is by an equation of the form

$$\mathbf{p}(t) \; = \; T\,M\,V, \qquad (*)$$

where $T = [t^3 \;\; t^2 \;\; t \;\; 1]$, M is a polynomial change-of-basis, and V is a column vector of ‘‘observations’’. Specifically, suppose M and V are

$$M \; = \; \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad V \; = \; \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}.$$

(a) [2] Write an expression for the derivative $\mathbf{p}'(t)$ of $\mathbf{p}$ in a form similar to Eq. (*).

(b) [8] Compute $\mathbf{p}(0)$, $\mathbf{p}(1)$, $\mathbf{p}'(0)$, $\mathbf{p}'(1)$. Show your work.

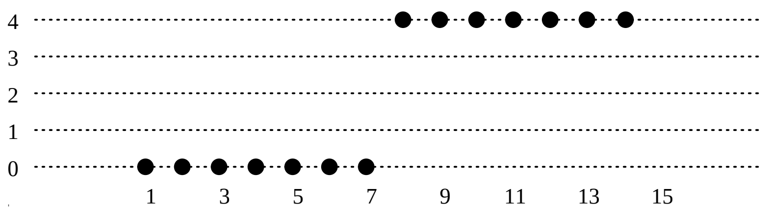
4. [12] Let $s = s_1, s_2, s_3, \cdots, s_n$ be a discrete signal. Let f be a filter, which is also a sequence of values

$$f = f_L, f_{L+1}, \cdots, f_{-1}, f_0, f_1, \cdots, f_{R-1}, f_R,$$

where $L < 0$ is the support of the filter to the left (i.e., into the past) and $R > 0$ is the support of the filter to the right (i.e., into the future). Assume that s_i is zero for any $i > n$ or $i \le 0$. To compute a filtered value $\overline{s}_i$, perform the following convolution:

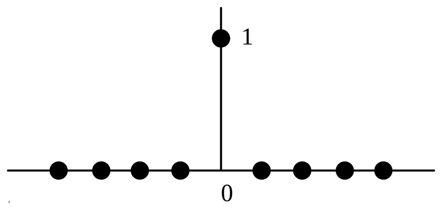
$$\overline{s}_i = \sum_{j=L}^{R} s_{i+j} f_j$$

for all i. In particular, our signal s is a step function given by

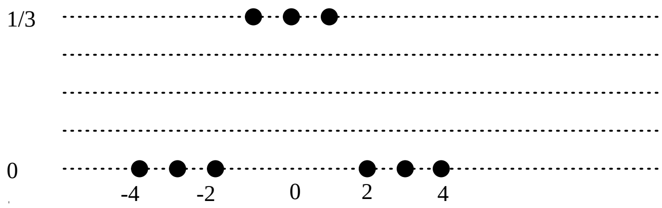


Convolve the following filters with s and plot the resulting signal in a form similar to the above signal. Then describe in words the effect of the filter on the original signal. The filter values are given beside the relevant point. Show some of your computations.

(a) [3]



(b) [4]



(c) [5]