**Fourier Series Example**

Let us compute the Fourier series for the function 

\[ f(x) = x \]

on the interval \([-\pi, \pi]\).

\(f\) is an odd function, so the \(a_n\) are zero, and thus the Fourier series will be of the form

\[ f(x) = \sum_{n=1}^{\infty} b_n \sin nx. \]

Furthermore, the \(b_n\) can be written in closed form.

Using integration by parts,

\[ b_n = \frac{2}{\pi} \int_0^{\pi} x \sin nx \, dx \]

\[ = \frac{2}{\pi n^2} (\sin nx - nx \cos nx) \bigg|_0^{\pi} \]

\[ = \frac{2}{\pi n^2} (\sin n\pi - n\pi \cos n\pi) \]

\[ = -\frac{2n\pi}{n^2 \pi} \cos n\pi \]

\[ = -\frac{2}{n} \cos n\pi. \]

\(\cos n\pi = -1\) when \(n\) is odd and \(\cos n\pi = 1\) when \(n\) is even.
Thus the above expression is equal to \(2/n\) when \(n\) is odd and \(-2/n\) when \(n\) is even.

Therefore our Fourier series is

\[
f(x) = 2\sin x - \sin 2x + \frac{2}{3}\sin 3x - \frac{2}{4}\sin 4x + \frac{2}{5}\sin 5x - \cdots.
\]
Approximate Spectrum

Magnitude of Spectrum
Another Example

Let $f$ be a box function defined on $[-\pi, \pi]$ as follows

$$f(x) = \begin{cases} 0 & \text{if } |x| > 1, \\ 1 & \text{if } |x| \leq 1. \end{cases}$$

This function contains two discontinuities.

We have arranged for this function to be an even function, so that its Fourier series is of the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx.$$  

Computing the $a_n$ is easy to do by hand if we simply observe that $f$ is nonzero only over $[-1, +1]$.

Thus

$$a_n = \frac{1}{\pi} \int_{-1}^{+1} f(x) \cos nx, \quad n = 0, 1, 2, \cdots$$

$$= \frac{1}{\pi} \int_{-1}^{+1} \cos nx$$

$$= \frac{\sin nx}{n\pi} \bigg|_{-1}^{+1}$$

$$= \frac{2}{n\pi} \sin n.$$
Observe that $\sin n / n \to 1$ as $n \to 0$.

Our Fourier series is therefore

$$f(x) = \frac{1}{\pi} \left( 1 + \sum_{n=1}^{\infty} \frac{2}{n} \sin n \cos nx \right).$$

Four terms:
20 terms:

50 terms:
The Fourier Transform

The Fourier transform of a function $f(x)$ is defined as

$$F(\omega) = \int_{-\infty}^{+\infty} f(x) e^{-i\omega x} \, dx,$$

and the inverse Fourier transform of $F(\omega)$ is

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\omega) e^{-i\omega x} \, d\omega,$$

where $i = \sqrt{-1}$.

$F$ is the spectrum of $f$.

When $f$ is even or odd, the Fourier transform reduces to the cosine or sine transform:

$$F_c(\omega) = \frac{2}{\pi} \int_{0}^{+\infty} f(x) \cos \omega x \, dx.$$

$$F_s(\omega) = \frac{2}{\pi} \int_{0}^{+\infty} f(x) \sin \omega x \, dx.$$

These latter two functions can be directly related to the $a_n$ and $b_n$ terms in a Fourier series.
Example: a line again

Let $f(x)$ be $x$ when $x \in [-\pi, \pi]$ and is zero outside this interval.

$f$ odd $\rightarrow f \cdot \cos$ is odd, and so

$$F_c(\omega) = 0.$$ 

On the other hand,

$$F_s(\omega) = \frac{2}{\pi} \int_0^{+\infty} x \sin \omega x \, dx$$

$$= 2 \frac{\sin x \omega - x \omega \cos x \omega}{\pi \omega^2} \bigg|_{x = 0}^{x = \pi}$$

$$= 2 \frac{\sin \pi \omega - \pi \omega \cos \pi \omega}{\pi \omega^2}.$$
Compare to Fourier series "spectrum".
Yet another example: a box again

\[
\text{box}(x) = \begin{cases} 0 & \text{if } |x| > 1. \\ 1 & \text{if } |x| \leq 1. \end{cases}
\]

Then

\[
F(\omega) = \int_{-\infty}^{+\infty} \text{box}(x) e^{-i\omega x} \, dx
\]

\[
= \int_{-1}^{1} e^{-i\omega x} \, dx
\]

\[
= i \frac{e^{-i\omega x}}{\omega} \bigg|_{x=-1}^{x=1}
\]

\[
= i \frac{e^{-i\omega}}{\omega} - i \frac{e^{i\omega}}{\omega}.
\]

Recalling that \( e^{-i\omega} = \cos \omega - i \sin \omega \), the cosine terms cancel out, and since \( i^2 = -1 \),

\[
F(\omega) = 2 \frac{\sin \omega}{\omega}
\]

\[
= 2\text{sinc}(\omega).
\]
So, \( \text{FT} (\text{box}) = \text{sinc} \), and the reverse is true too.

Furthermore, if

\[
\text{box}_k(x) = \begin{cases} 
0 & \text{if } |x| > k. \\
1 & \text{if } |x| \leq k.
\end{cases}
\]

Then it is easy to see that

\[
F(\omega) = 2 \frac{\sin(k\omega)}{\omega}.
\]
Important Theorems

Suppose $F(\omega) = \mathbf{FT}(f)$ and $G(\omega) = \mathbf{FT}(g)$ are the spectra (i.e., Fourier transforms) of $f$ and $g$, respectively, assuming they exist.

The convolution theorem states that

$$\mathbf{FT}(f * g) = FG.$$ 

In other words, convolution in spatial domain is equivalent to multiplication in frequency domain.

The analogous theorem called the modulation theorem expresses the duality of the converse operations:

$$\mathbf{FT}(fg) = \frac{1}{2\pi} (F*G).$$

We therefore have a duality: multiplication of functions in one domain is equivalent under the Fourier transform to convolution in the other.
A low-pass filter $h$ is one that, under a convolution with any function $f$, admits only the frequencies of $f$ that fall within a specific bandwidth (i.e., frequency interval) $[-\omega_h, \omega_h]$.

What must the shape of $h$ be in frequency domain? I.e., what is $\text{FT}(h)$?

What must the shape of $h$ be in spatial domain?

There is only one ideal (family of) low-pass filter for 1-D signals.

How many classes of ideal low-pass filters are there in 2-D?
The only ideal low-pass filter is a sinc in spatial domain or box in frequency domain.

The effect, in frequency domain, of spatial filtering using a sinc filter.
The Sampling Theorem Revisited

Let $f(t)$ be a band-limited signal. Specifically, let the spectrum $F(\omega)$ of $f(t)$ be such that $F(\omega) = 0$ for $|\omega| > \omega_m$, for some “maximum frequency” $\omega_m > 0$.

Let $\Delta t$ be the spacing at which we take samples of $f(t)$. Furthermore, we define the circular *sampling rate* $\omega_s$ as

$$\omega_s = \frac{2\pi}{\Delta t}.$$ 

Then $f(t)$ can be uniquely represented by a sequence of samples $f(i\Delta t), \; i \in \mathbb{Z}$ if

$$\omega_s > 2\omega_m.$$ 

I.e., our sampling rate must exceed twice the maximum frequency of the function.
**Important Fact 1:**

\[ f(x) * \delta(x-a) = f(x-a). \]

Convolution with \( \delta \) creates a copy of \( f \) shifted by \( a \) units.

**Important Fact 2:**

Let \( \delta_\varepsilon(x) \) denote a box of half-width \( \varepsilon \) and of area one centred at position \( x = 0 \).

Let \( f(x) \) be a function that is smooth around \([-\varepsilon, +\varepsilon]\).

Then

\[ f(x) \delta_\varepsilon(x) \approx f(0) \delta_\varepsilon(x). \]

As \( \varepsilon \to 0 \), \( \delta_\varepsilon(x) \to \delta(x) \),

\[ f(x) \delta(x) \approx f(0) \delta(x). \]

This multiplication, has the effect of *sampling* \( f \) at \( x = 0 \).

More generally,

\[ f(x) \delta(x-a) \approx f(a) \delta(x-a), \]

for an arbitrary \( a \in \mathbb{R} \).

Thus outside of an integral sign, \( \delta \) works as a sampling operator.
**Sampling Train**

Visualise an infinite sequence of “impulses” or δ-functions, with one impulse placed at each sampling position \( i \Delta T \) as in

We can define this sampling train or “comb” of impulses as

\[
s(t) = \sum_{i=-\infty}^{+\infty} \delta(t - i \Delta t).
\]

The summation can be thought of the glue that holds a sequence of impulses together, and because \( \Delta t > 0 \), the impulses are spaced so that they do not overlap.
**Sampling Operation**

We saw that we could effectively sample a function $f$ at a any desired position $a$ by placing a $\delta$-function at $a$ and multiplying it with $f$.

Therefore, multiplying $f$ with $s$ takes samples of $f$ at our desired positions:

$$f_s = f s = \sum_{i = -\infty}^{+\infty} f(i\Delta t) \delta(t - i\Delta t).$$

This new train of “scaled” impulses is:
Basic Argument

Suppose $f$ and $s$ have spectra $F$ and $S$, respectively. Since $f_s$ is a product of two functions $f$ and $s$, then the modulation theorem states that its Fourier transform is a convolution:

$$\text{FT}(f_s) = F_s(\omega) = \frac{1}{2\pi} F(\omega) * S(\omega).$$

For our purposes $f$, and therefore $F$, is an arbitrary function. However, we can compute the spectrum of $s$.

It indeed turns out that the spectrum of a train of impulses of spacing $\Delta t$ is another train of impulses in frequency domain with spacing $2\pi/\Delta t$, which we defined above to be $\omega_s$. Formally,

$$\text{FT}(s) = S(\omega) = \frac{2\pi}{\Delta t} \sum_{k=-\infty}^{+\infty} \delta(\omega - k\omega_s).$$

We can put this back into our expression for $F_s$:

$$F_s = \frac{1}{2\pi} F(\omega) * S(\omega)$$

$$= \frac{1}{2\pi} \cdot \frac{2\pi}{\Delta t} F(\omega) * \left( \sum_{k=-\infty}^{+\infty} \delta(\omega - k\omega_s) \right)$$

$$= \frac{1}{\Delta t} \sum_{k=-\infty}^{+\infty} F(\omega - k\omega_s).$$
The Argument as a Picture

We need to prevent overlap of the spectra, for otherwise we’d have no hope of extracting a single spectrum.

Therefore,

\[ \omega_m < \omega_s - \omega_m. \]

This implies that

\[ \omega_s > 2\omega_m, \]

which establishes the theorem.
How do we get the signal back to the real world?

We use a box filter in frequency domain to extract one copy of the spectrum of $F$ from $F_s$:

$$F(\omega) = F_s(\omega) B(\omega),$$

and by the convolution theorem, we can reconstruct $f$ by convolution:

$$f(t) = f_s(t) * \text{sinc}_B(t),$$

where $\text{sinc}_B(t)$ is the inverse Fourier transform of $B$.

**Exercise:** Suppose our box $B$ is to have width $\omega_b$ and height $\Delta t$. Then show that

$$\text{sinc}_B(t) = \frac{\Delta t \omega_b}{\pi} \text{sinc} \left( \frac{\omega_b t}{\pi} \right).$$

So a sinc is both an ideal low-pass filter AND an ideal reconstruction function (i.e., interpolant).
Analytic Filtering

One possibility: to filter a signal $s$ with a filter $f$, rather than compute a convolution, we instead:

- compute Fourier transforms $S$ and $F$.
- compute $SF$.
- take the inverse Fourier transform.

Sometimes this even works!

```plaintext
# Analytic filtering of signal s with filter f in frequency domain.

filter := proc(s,f,x)
    local S,F,SF,sf,w;
    S := evalc(fourier(s,x,w));
    F := evalc(fourier(f,x,w));
    SF := S*F: # NOTE: regular multiplication
    sf := evalc(invfourier(SF,w,x));
end:
```
Analytically Filtering a Polynomial

> p;
(x-2) (x-1.5) (x-1) (x-.5) x (x+.5) (x+1) (x+1.5) (x+2)

> sort(expand(p),x);
9 7 5 3
x - 7.50 x + 17.0625 x - 12.8125 x + 2.2500 x

We can filter \( p \) as in the following Maple session.

> Digits := 5: # keep output size manageable
> gauss := 1/(sqrt(2*Pi)*s)*exp(-x^2/(2*s^2)):

> pg := filter(p,gauss,x):

> collect(sort(collect(pg,x),s),Pi); # make output more readable

9
(3.1416 x 2 7 4 2 5
+ (113.10 s - 23.562) x + (53.605 + 1187.5 s - 494.80 s ) x 2 6 4 3
+ (536.05 s - 40.252 + 3958.4 s - 2474.0 s ) x
2 6 4 8
+ (7.0685 - 120.76 s - 2474.0 s + 804.10 s + 2968.8 s ) x)/Pi
Graphically, varying the standard deviation
Summary

If we know that a signal is bandlimited (and we know what that limit is), then we have a lower bound for the minimum sampling density.

If the signal is not bandlimited, we can prefilter it into one that is. Then we can compute the right sampling rate.

But Is It Practical?

In a word, mostly no, sometimes yes, and occasionally, maybe.