Fourier Series Example

Let us compute the Fourier series for the function

$$f(x) = x$$

on the interval $[-\pi, \pi]$.

f is an odd function, so the a_n are
zero, and thus the Fourier series will be of the form

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx.$$

Furthermore, the b_n can be written in closed form.

Using integration by parts,

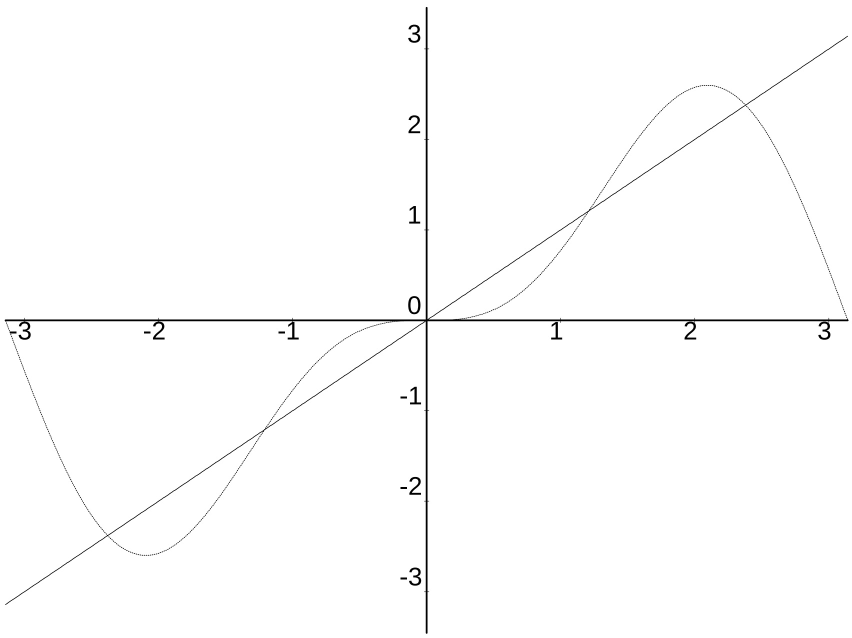
$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi} x \sin nx \, dx \\ &= \frac{2}{\pi n^2}(\sin nx - nx\cos nx)\Big|_0^{\pi} \\ &= \frac{2}{\pi n^2}(\sin n\pi - n\pi\cos n\pi) \\ &= -\frac{2n\pi}{n^2\pi}\cos n\pi \\ &= -\frac{2}{n}\cos n\pi. \end{aligned}$$

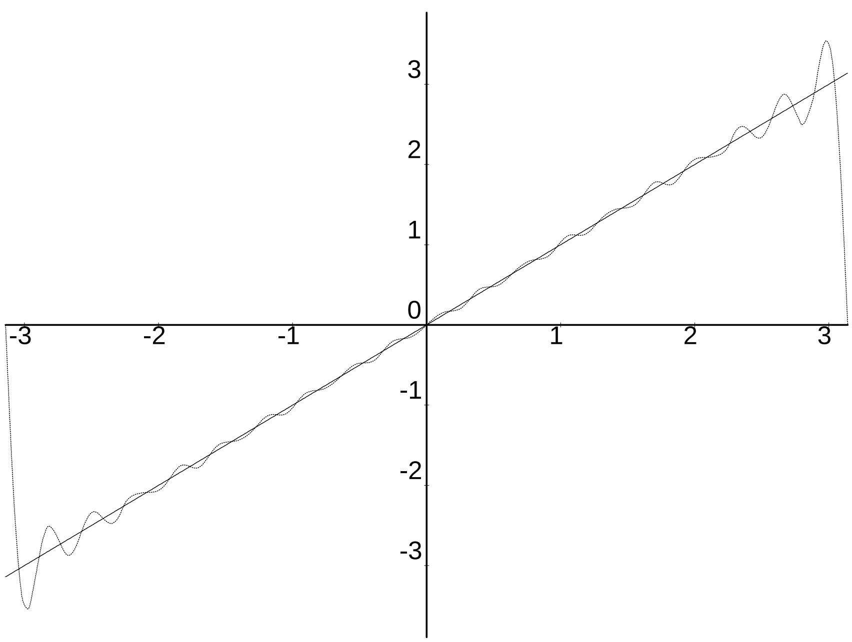
$\cos n\pi = -1$ when n is odd and $\cos n\pi = 1$ when n is even.

Thus the above expression is equal to $2/n$ when n is odd and $-2/n$ when n is even.

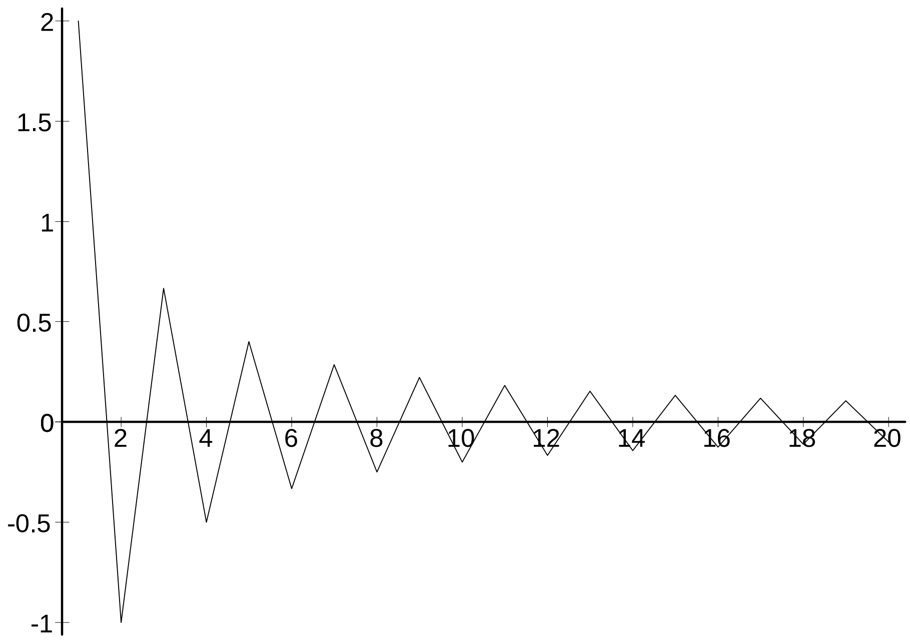
Therefore our Fourier series is

$$f(x) = 2\sin x - \sin 2x + \frac{2}{3}\sin 3x - \frac{2}{4}\sin 4x + \frac{2}{5}\sin 5x - \cdots .$$

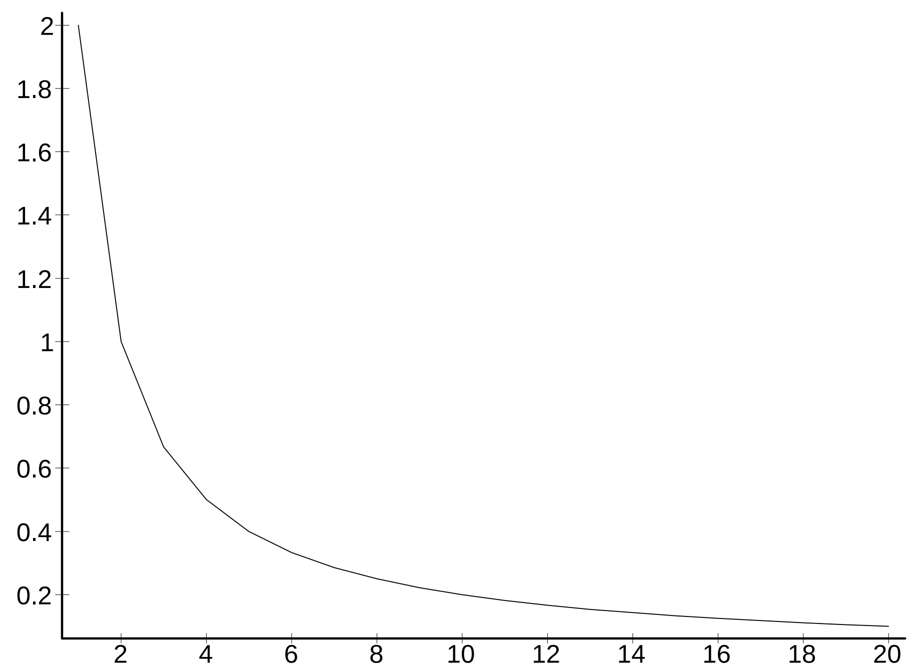




Approximate Spectrum



Magnitude of Spectrum



Another Example

let f be a *box* function defined on $[-\pi,\pi]$ as follows

$$f(x) = \begin{cases} 0 & \text{if } |x| > 1. \\ 1 & \text{if } |x| \le 1. \end{cases}$$

This function contains two discontinuities.

We have arranged for this function to be an even function, so that its Fourier series is of the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx.$$

Computing the a_n is easy to do by hand if we simply observe that f is nonzero only over $[-1,+1]$.

Thus

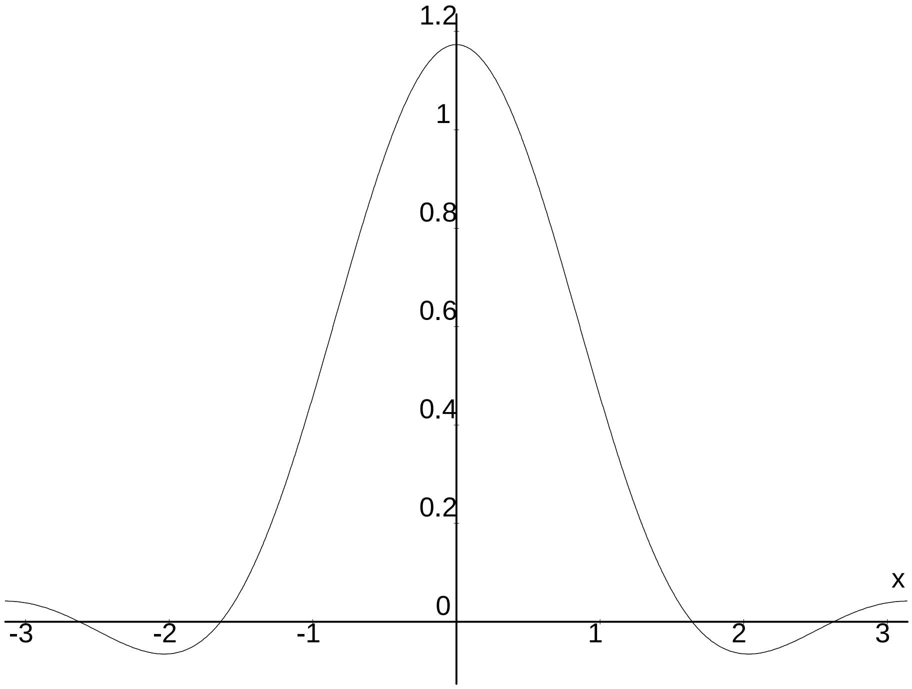
$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-1}^{+1} f(x)\cos nx, \quad n = 0, 1, 2, \cdots \\ &= \frac{1}{\pi}\int_{-1}^{+1} \cos\, nx \\ &= \frac{\sin nx}{n\pi}\Big|_{-1}^{+1} \\ &= \frac{2}{n\pi}\sin n. \end{aligned}$$

Observe that $\sin n / n \rightarrow 1$ as $n \rightarrow 0$.

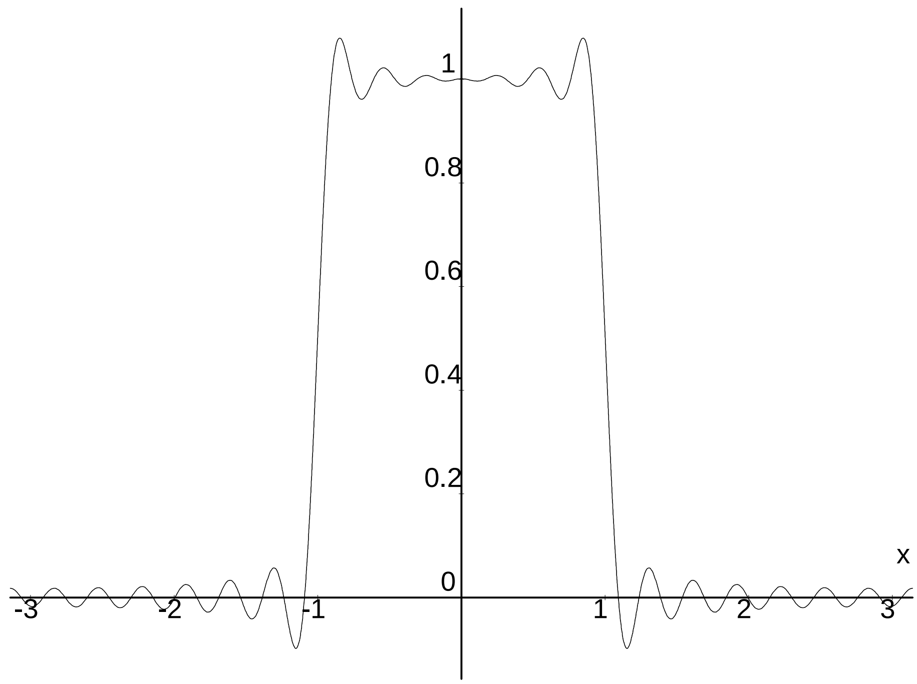
Our Fourier series is therefore

$$f(x) = \frac{1}{\pi}\left(1 + \sum_{n=1}^{\infty} \frac{2}{n} \sin n \cos nx\right).$$

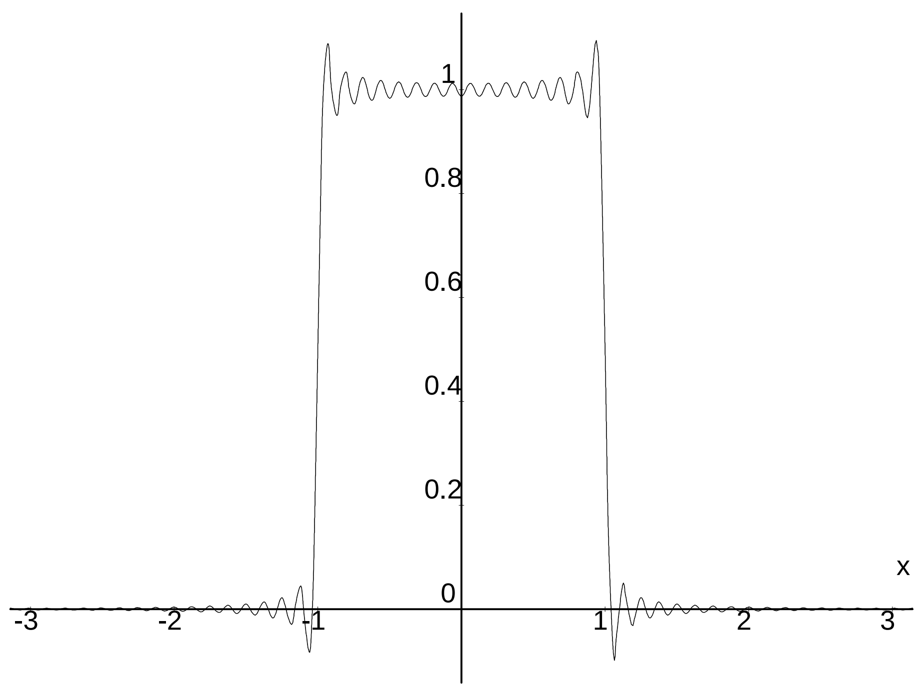
Four terms:



20 terms:



50 terms:



The Fourier Transform

The *Fourier transform* of a function $f(x)$ is defined as

$$F(\omega) \;=\; \int_{-\infty}^{+\infty} f(x)\, e^{-i\omega x}\, dx,$$

and the *inverse Fourier transform* of $F(\omega)$ is

$$f(x) \;=\; \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\omega)\, e^{-i\omega x}\, d\omega,$$

where $i = \sqrt{-1}$.

F is the *spectrum* of f.

When f is even or odd, the Fourier transform reduces to the *cosine* or *sine* transform:

$$F_c(\omega) \;=\; \frac{2}{\pi} \int_{0}^{+\infty} f(x) \cos \omega x\, dx.$$

$$F_s(\omega) \;=\; \frac{2}{\pi} \int_{0}^{+\infty} f(x) \sin \omega x\, dx.$$

These latter two functions can be directly related to the a_n and b_n terms in a Fourier series.

Example: a line again

Let $f(x)$ be x when $x \in [-\pi, \pi]$ and is zero outside this interval.

f odd $\rightarrow f \cdot \cos$ is odd, and so

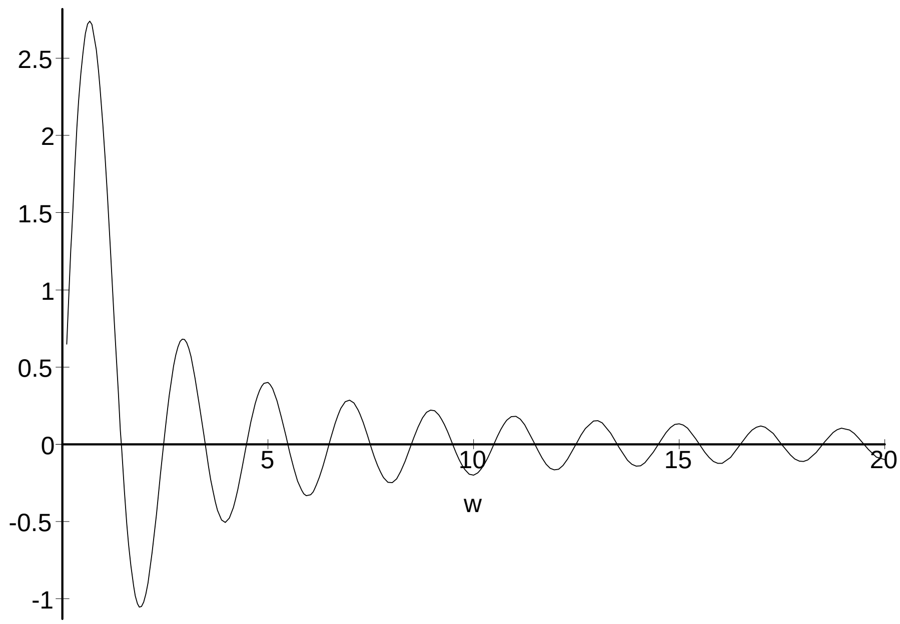
$$F_c(\omega) = 0.$$

On the other hand,

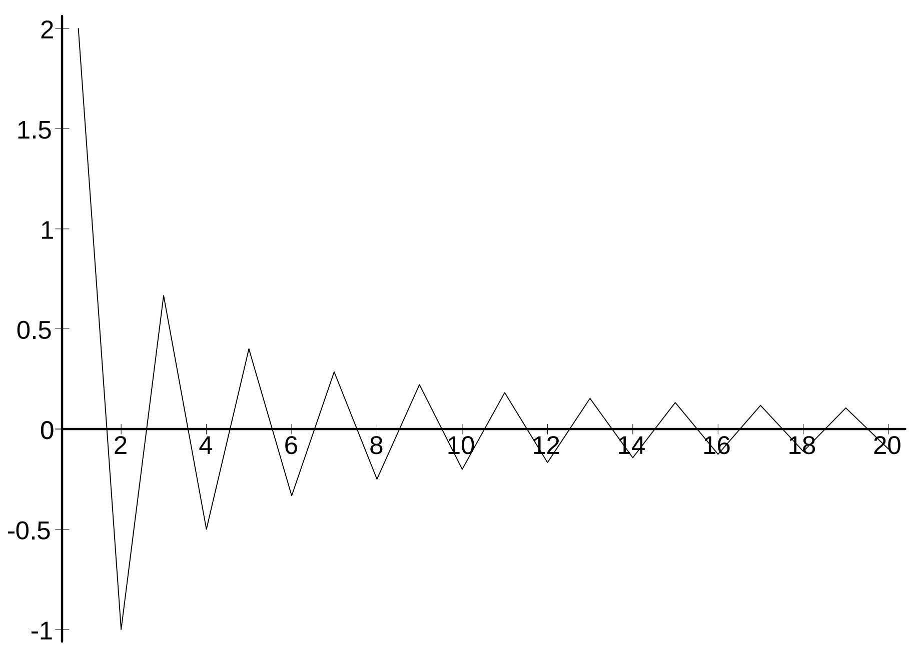
$$F_s(\omega) = \frac{2}{\pi} \int_0^{+\infty} x \sin \omega x \, dx$$

$$= 2 \frac{\sin x\omega - x\omega \cos x\omega}{\pi\omega^2} \Big|_{x=0}^{x=\pi}$$

$$= 2 \frac{\sin \pi\omega - \pi\omega \cos \pi\omega}{\pi\omega^2}.$$



Compare to Fourier series "spectrum".



Yet another example: a box again

$$\mathrm{box}(x) = \begin{cases} 0 & \text{if } |x| > 1. \\ 1 & \text{if } |x| \le 1. \end{cases}$$

Then

$$F(\omega) = \int_{-\infty}^{+\infty} \mathrm{box}(x)\, e^{-i\omega x}\, dx$$

$$= \int_{-1}^{1} e^{-i\omega x}\, dx$$

$$= i\,\frac{e^{-i\omega x}}{\omega}\Bigg|_{x=-1}^{x=1}$$

$$= i\,\frac{e^{-i\omega}}{\omega} - i\,\frac{e^{i\omega}}{\omega}.$$

Recalling that $e^{-i\omega} = \cos\omega - i\sin\omega$, the cosine terms cancel out, and since $i^2 = -1$,

$$F(\omega) = 2\frac{\sin\omega}{\omega}$$

$$= 2\mathrm{sinc}(\omega).$$

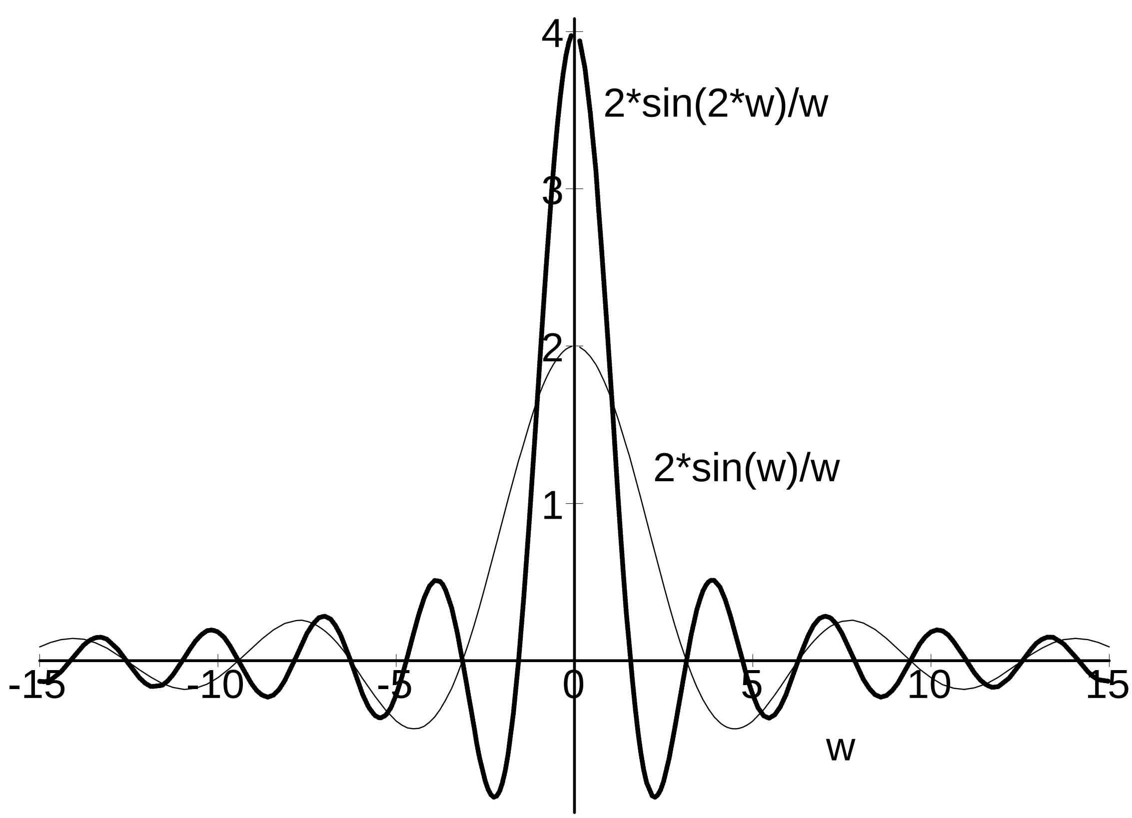
So, **FT**(box) = sinc, and the reverse is true too..

Furthermore, if

$$\mathrm{box}_k(x) = \begin{cases} 0 & \text{if } |x| > k. \\ 1 & \text{if } |x| \leq k. \end{cases}$$

Then it is easy to see that

$$F(\omega) = 2\frac{\sin(k\omega)}{\omega}.$$



Important Theorems

Suppose $F(\omega) = \mathbf{FT}(f)$ and $G(\omega) = \mathbf{FT}(g)$ are the spectra (i.e., Fourier transforms) of f and g, respectively, assuming they exist.

The *convolution theorem* states that

$$\mathbf{FT}(f * g) \quad = \quad F\, G.$$

In other words, convolution in spatial domain is equivalent to multiplication in frequency domain.

The analogous theorem called the *modulation theorem* expresses the duality of the converse operations:

$$\mathbf{FT}(f\, g) \quad = \quad \frac{1}{2\pi}\,(F * G)\,.$$

We therefore have a duality: multiplication of functions in one domain is equivalent under the Fourier transform to convolution in the other.

A *low-pass* filter h is one that, under a convolution with any function f, admits only the frequencies of f that fall within a specific bandwidth (i.e., frequency interval) $[-\omega_h, \omega_h]$.

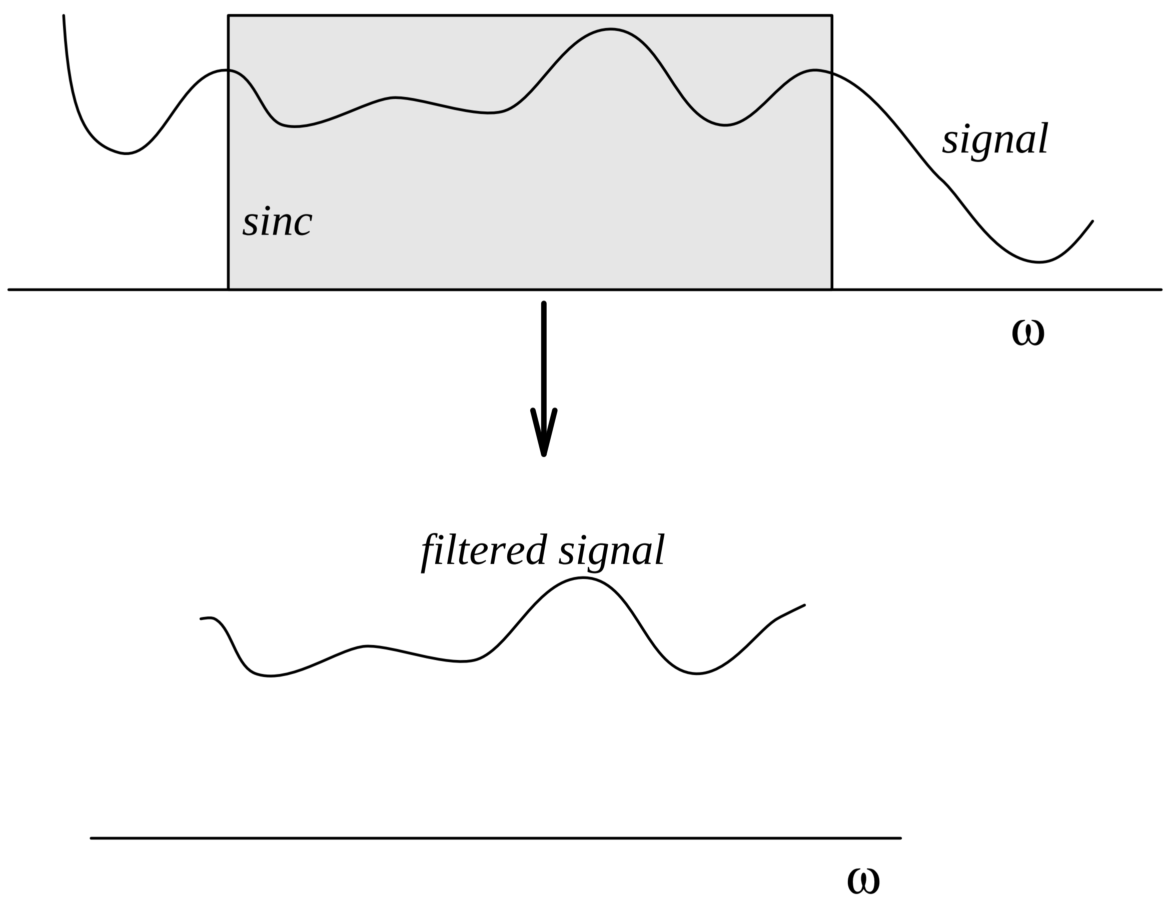
What must the shape of h be in frequency domain? I.e., what is **FT**(h)?

What must the shape of h be in spatial domain?

There is only one ideal (family of) low-pass filter for 1-D signals.

How many classes of ideal low-pass filters are there in 2-D?

The only ideal low-pass filter is a sinc in spatial domain or box in frequency domain.



The effect, in frequency domain, of spatial filtering using a sinc filter.

The Sampling Theorem Revisited

Let $f(t)$ be a band-limited signal. Specifically, let the spectrum $F(\omega)$ of $f(t)$ be such that $F(\omega) = 0$ for $|\omega| > \omega_m$, for some ''maximum frequency'' $\omega_m > 0$.

Let Δt be the spacing at which we take samples of $f(t)$. Furthermore, we define the circular *sampling rate* ω_s as

$$\omega_s = \frac{2\pi}{\Delta t} .$$

Then $f(t)$ can be uniquely represented by a sequence of samples $f(i\Delta t),\ i \in$ **Z** if

$$\omega_s > 2\omega_m .$$

I.e., our sampling rate must exceed twice the maximum frequency of the function.

Important Fact 1:

$$f(x) * \delta(x-a) \quad = \quad f(x-a).$$

Convolution with δ creates a copy of f shifted by a units.

Important Fact 2:

Let $\delta_\varepsilon(x)$ denote a box of half-width ε and of area one centred at position $x = 0$.

Let $f(x)$ be a function that is smooth around $[-\varepsilon, +\varepsilon]$.

Then

$$f(x)\,\delta_\varepsilon(x) \quad \approx \quad f(0)\,\delta_\varepsilon(x).$$

As $\varepsilon \to 0$, $\delta_\varepsilon(x) \to \delta(x)$,

$$f(x)\,\delta(x) \quad \approx \quad f(0)\,\delta(x).$$

This multiplication, has the effect of *sampling* f at $x = 0$.

More generally,

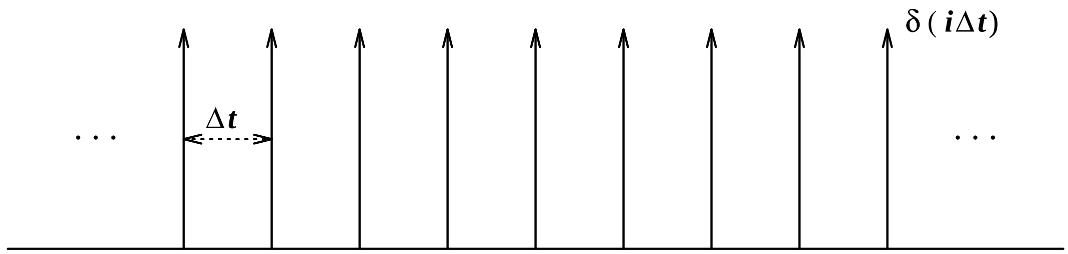
$$f(x)\,\delta(x-a) \quad \approx \quad f(a)\,\delta(x-a),$$

for an arbitrary $a \in \mathbf{R}$.

Thus outside of an integral sign, δ works as a sampling operator.

Sampling Train

Visualise an infinite sequence of ‘‘impulses’’ or δ-functions, with one impulse placed at each sampling position $i\Delta T$ as in



We can define this sampling train or ‘‘comb’’ of impulses as

$$s(t) \;=\; \sum_{i=-\infty}^{+\infty} \delta(t - i\Delta t).$$

The summation can be thought of the glue that holds a sequence of impulses together, and because $\Delta t > 0$, the impulses are spaced so that they do not overlap.

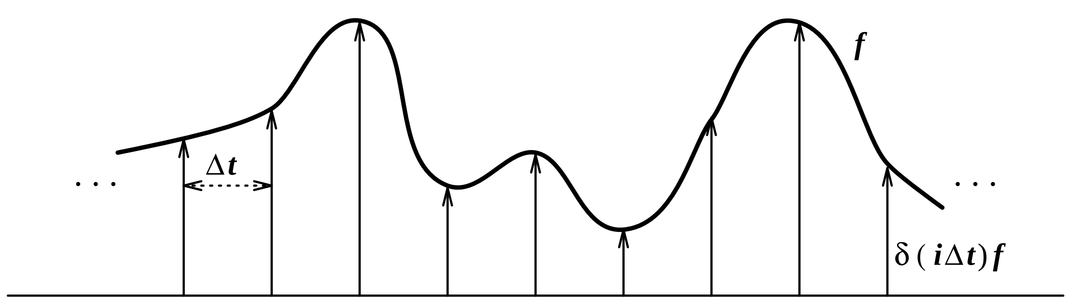
Sampling Operation

We saw that we could effectively sample a function f at a any desired position a by placing a δ-function at a and multiplying it with f.

Therefore, multiplying f with s takes samples of f at our desired positions:

$$f_S \;=\; f\,s \;=\; \sum_{i=-\infty}^{+\infty} f(i\Delta t)\,\delta(t - i\Delta t).$$

This new train of ‘‘scaled’’ impulses is:



Basic Argument

Suppose f and s have spectra F and S, respectively. Since f_s is a product of two functions f and s, then the modulation theorem states that its Fourier transform is a convolution:

$$\mathbf{FT}(f_s) \quad = \quad F_s(\omega) \quad = \quad \frac{1}{2\pi} F(\omega) * S(\omega).$$

For our purposes f, and therefore F, is an arbitrary function. However, we can compute the spectrum of s.

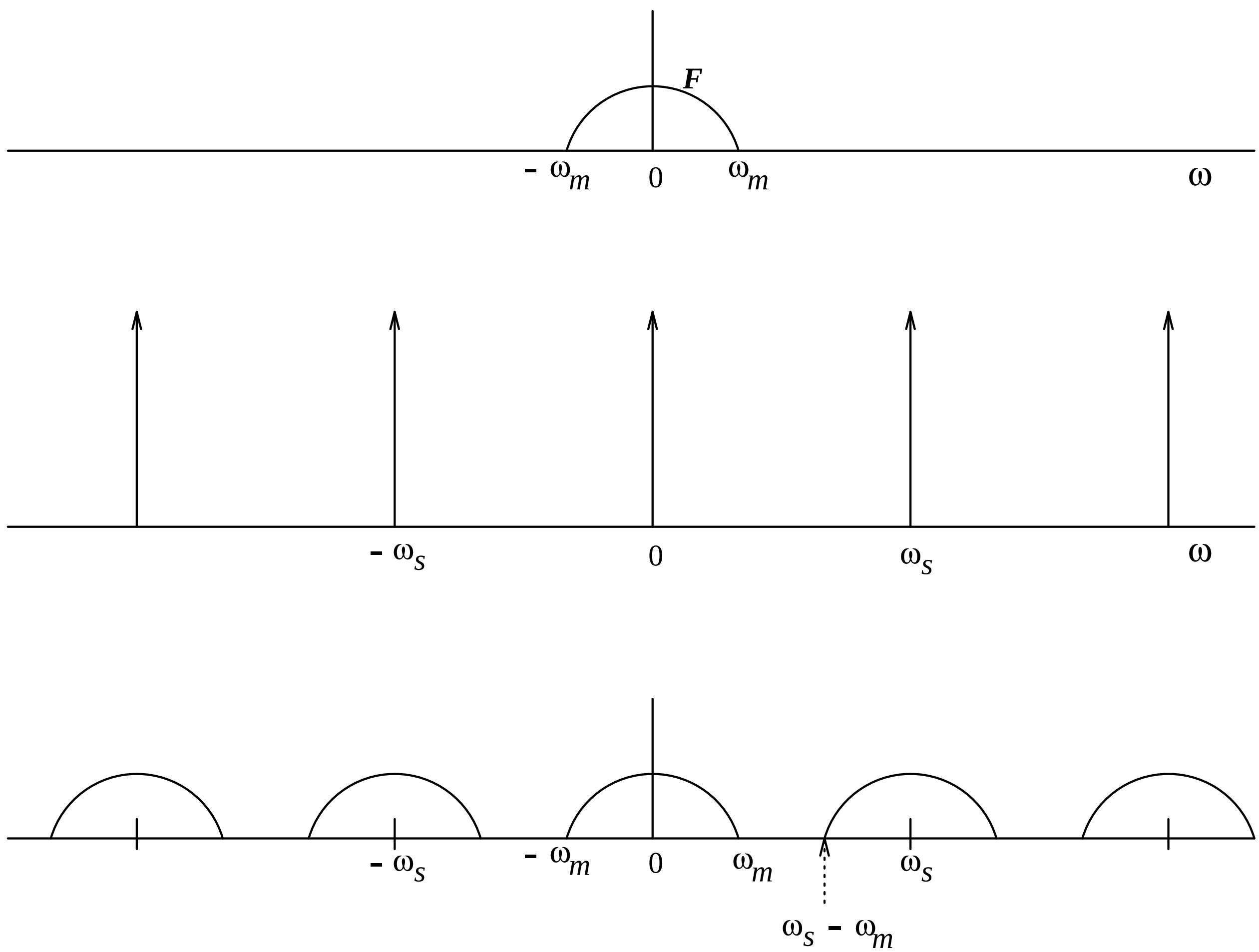
It indeed turns out that the spectrum of a train of impulses of spacing Δt is *another* train of impulses in frequency domain with spacing $2\pi/\Delta t$, which we defined above to be ω_s. Formally,

$$\mathbf{FT}(s) \quad = \quad S(\omega) \quad = \quad \frac{2\pi}{\Delta t} \sum_{k=-\infty}^{+\infty} \delta(\omega - k\omega_s).$$

We can put this back into our expression for F_s:

$$\begin{aligned} F_s \quad &= \quad \frac{1}{2\pi} F(\omega) * S(\omega) \\ &= \quad \frac{1}{2\pi} \cdot \frac{2\pi}{\Delta t} \; F(\omega) * \left(\sum_{k=-\infty}^{+\infty} \delta(\omega - k\omega_s) \right) \\ &= \quad \frac{1}{\Delta t} \sum_{k=-\infty}^{+\infty} F(\omega - k\omega_s). \end{aligned}$$

The Argument as a Picture



We need to prevent overlap of the spectra, for otherwise we'd have no hope of extracting a single spectrum.

Therefore,

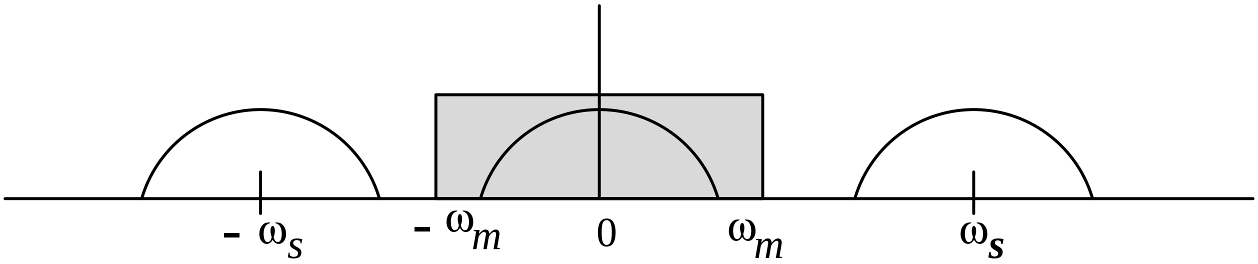
$$\omega_m < \omega_s - \omega_m .$$

This implies that

$$\omega_s > 2\omega_m ,$$

which establishes the theorem.

How do we get the signal back to the real world?



We use a box filter in frequency domain to extract one copy of the spectrum of F from F_s:

$$F(\omega) = F_s(\omega)\,B(\omega),$$

and by the convolution theorem, we can reconstruct f by convolution:

$$f(t) = f_s(t) * \mathrm{sinc}_B(t),$$

where $\mathrm{sinc}_B(t)$ is the inverse Fourier transform of B.

Exercise: Suppose our box B is to have width ω_b and height Δt. Then show that

$$\mathrm{sinc}_B(t) = \frac{\Delta t\,\omega_b}{\pi}\,\mathrm{sinc}\left(\frac{\omega_b\,t}{\pi}\right).$$

So a sinc is both an ideal low-pass filter AND an ideal reconstruction function (i.e., interpolant).

Analytic Filtering

One possibility: to filter a signal s with a filter f, rather than compute a convolution, we instead:

- compute Fourier transforms S and F.
- compute SF.
- take the inverse Fourier transform.

Sometimes this even works!

```
#
# Analytic filtering of signal s with filter f in
# frequency domain.
#
filter := proc(s,f,x)
    local S,F,SF,sf,w;
    S := evalc(fourier(s,x,w));
    F := evalc(fourier(f,x,w)):
    SF := S*F:           # NOTE: regular multiplication
    sf := evalc(invfourier(SF,w,x));
end:
```

Analytically Filtering a Polynomial

```
> p;
     (x-2) (x-1.5) (x-1) (x-.5) x (x+.5) (x+1) (x+1.5) (x+2)

> sort(expand(p),x);
       9          7             5             3
      x  - 7.50 x  + 17.0625 x  - 12.8125 x  + 2.2500 x
```

We can filter p as in the following Maple session.

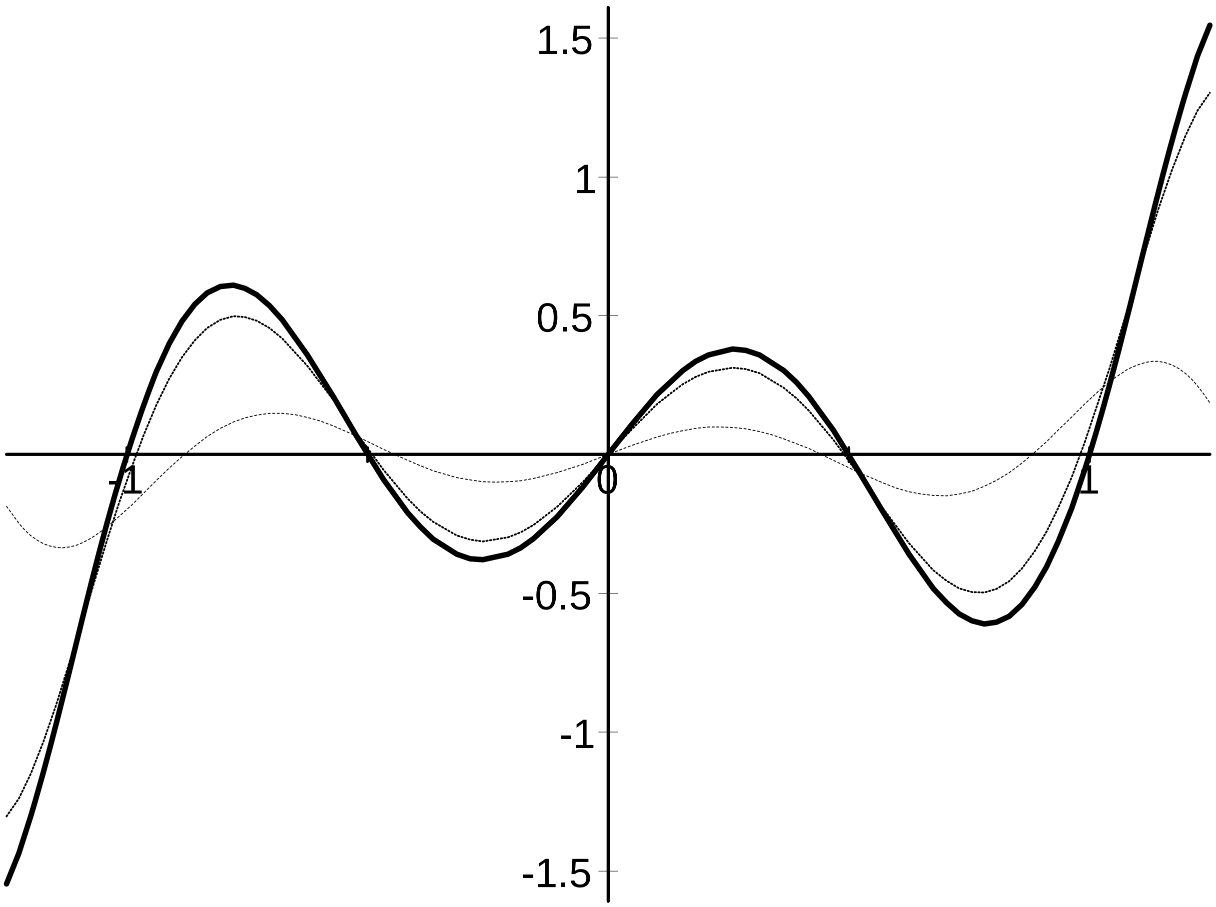
```
> Digits := 5:                   # keep output size manageable
> gauss := 1/(sqrt(2*Pi)*s)*exp(-x^2/(2*s^2)):

> pg := filter(p,gauss,x):

> collect(sort(collect(pg,x),s),Pi);  # make output more readabl

         9
(3.1416 x
          2            7                     4          2   5
+ (113.10 s  - 23.562) x  + (53.605 + 1187.5 s  - 494.80 s ) x
          2                  6           4   3
+ (536.05 s  - 40.252 + 3958.4 s  - 2474.0 s ) x
                   2           6           4           8
+ (7.0685 - 120.76 s  - 2474.0 s  + 804.10 s  + 2968.8 s ) x)/Pi
```

Graphically, varying the standard deviation



Summary

If we know that a signal is bandlimited (and we know what that limit is), then we have a lower bound for the minimum sampling density.

If the signal is *not* bandlimited, we can prefilter it into one that is. Then we can compute the right sampling rate.

But Is It Practical?

In a word, mostly no, sometimes yes, and occasionally, maybe.