Chapter 3

Particle Volumes

In order to develop a new model for shape representation and manipulation, we construct synthetic materials that exhibit useful physical and geometric properties. We do not try to simulate a given material accurately, but rather to imitate properties that we find useful for sculpting, animation, and reconstruction tasks. The ability to create large changes in the geometric structure and the ability to change the topology of the shape at will, suggests the use of primitive shape elements, where individual elements do not enforce a given topology, but rather the topology is an emergent property of the the self-organizing system. As such, we have chosen to use an interacting particle system that moves in accordance with the laws of Newtonian dynamics. The physical properties of our system follows from our selection of inter-particle potential energy functions. The geometry is derived from the relative positions of the particles.

3.1 Particle Systems

Each particle represents a primitive element with mass, volume, and physical properties defined between particles. A particle system is defined as a collection of point masses, where each particle has a position and mass, and moves under the influences of forces according to the principles of classical physics. Such a system is governed by the set of ordinary differential equations of motion

$$m_i\ddot{\mathbf{x}}_i + \gamma_i\dot{\mathbf{x}}_i + \mathbf{f}_i^{\mathtt{int}} = \mathbf{f}_i^{\mathtt{ext}} \qquad i = 1, \ldots, N, \tag{3.1}$$

where the subscript i denotes an attribute of particle i, N is the number of particles in the system, over struck dots denote time derivatives, m_i is its mass, $\ddot{\mathbf{x}}_i$ is its acceleration, $\dot{\mathbf{x}}_i$ is its velocity, γ_i is a damping coefficient that controls the rate of dissipation of the particle's kinetic energy, $\mathbf{f}_i^{\mathtt{int}}$ is the sum of inter-particle forces and $\mathbf{f}_i^{\mathtt{ext}}$ is the sum of external forces acting on the particle. Inter-particle force terms are functions of the form

$$\mathbf{f}_i^{\mathtt{int}}(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_N) \tag{3.2}$$

External forces are functions of single particle state and external state variables

$$\mathbf{f}_i^{\mathtt{ext}}(\mathbf{x}_i, \mathbf{S}), \tag{3.3}$$

where $\mathbf{S}$ is a set of external state variables such as gravity, the positions of obstacles or shaping tools, and volumetric data sets. A particle system can be tailored to a specific application by choosing appropriate inter-particle and external forces.

3.1.1 Potential Energy

Potential energy functions provide an elegant method of describing inter-particle force functions based on particle positions. For a potential function ϕ, the force exerted on particle i with position $\mathbf{x}_i$,

$$\mathbf{f}_i = -\nabla_{\mathbf{x}_i}\phi$$

is due to the gradient of the potential energy ϕ

$$\nabla_{\mathbf{x}_i}\phi = \left(\frac{\partial\phi}{\partial\mathbf{x}_{1_i}}, \frac{\partial\phi}{\partial\mathbf{x}_{2_i}}, \frac{\partial\phi}{\partial\mathbf{x}_{3_i}}\right)$$

with respect to the change in position.

A common practice, and one this dissertation adopts, is to define a particle's potential energy based on the pairwise additivity assumption. This assumption states that the total potential energy of a particle is the sum of the pairwise potential energies between that particle and every other particle. That is, the potential energy ϕ_i of particle i with respect to a system of N particles is given by

$$\phi_i = \sum_{j\neq i}^{N} \phi_{ij}, \tag{3.4}$$

where ϕ_{ij} is the potential energy between particles i and j, and N is the number of particles in the system. Thus the net inter-particle forces acting on a particle i due to the potential energy function is

$$\mathbf{f}_i = -\nabla_{\mathbf{x}_i}\phi_i = -\sum_{j\neq i}^{N} \nabla_{\mathbf{x}_i}\phi_{ij}. \tag{3.5}$$

The total potential energy of the system is the sum of the pairwise potential energies for all pairs of particles

$$E_P = \sum_{i}^{N} \phi_i = \sum_{i}^{N}\sum_{j\neq i}^{N} \phi_{ij}. \tag{3.6}$$

3.1.2 Kinetic Energy

The kinetic energy of the system is a measure of particle movement. The kinetic energy K_i of a single particle i is given by

$$K_i = \frac{1}{2}m_i v_i^2 = \frac{1}{2}m_i\|\dot{\mathbf{x}}_i\| \tag{3.7}$$

where m_i is the mass of the particle, and v_i is the scalar speed of the particle. The kinetic energy of the system is the sum of the individual particle kinetic energies

$$E_K = \sum_i^N K_i. \tag{3.8}$$

3.1.3 System Energy

The total energy of the system, E_S, is simply a summation of the individual particles' kinetic and potential energies,

$$E_S = E_K + E_P = \sum_i^N K_i + \sum_i^N \sum_{j \neq i}^N \phi_{ij}. \tag{3.9}$$

Systems whose dynamics are governed by potential functions and damping will evolve towards lower energy states. Differentiating the potential energy functions results in forces acting on the particles. These forces move the particles over time resulting in an increase in kinetic energy. Assuming no external forces, eventually the system will come to rest as the kinetic energy is dissipated by velocity based damping.

3.2 Dynamic Coupling

The equations of potential energy from the previous section provide a generalized framework from which to work, yet for practical purposes they are problematic. To compute all inter-particle forces, without any restricting assumptions on the potential energies, requires N^2 force computations. For a system with thousands of particles, this is clearly a problem. Since we are not concerned with the long range effects between particles, as astrophysicists are when studying the evolution of a galaxy, we instead choose to limit the inter-particle interactions so that only neighboring particles interact. We define neighboring particles to be pairs of particles that are closer than a specified neighborhood distance. Thus we can reduce force computations to $O(N \log N)$ for neighbor finding and $O(N)$ to compute the forces. In Chapter 7 we discuss in more detail the choice to limit particle interactions, how we compute the nearest neighbors, and how we derive the complexity measures.

Letting $\mathcal{N}_i$ be the set of particles that neighbor particle i, we can rewrite the force and energy equations as follows. Particle i's inter-particle potential energy (3.4) is rewritten as

$$\phi_i = \sum_{j \in \mathcal{N}_i}^N \phi_{ij}, \tag{3.10}$$

and the corresponding inter-particle force (3.5) is rewritten as

$$\mathbf{f}_i = -\nabla_{\mathbf{x}_i} \phi_i = -\sum_{j \in \mathcal{N}_i}^N \nabla_{\mathbf{x}_i} \phi_{ij}. \tag{3.11}$$

The kinetic energy equations (3.7) and (3.8) are unchanged. The total potential energy of the system (3.6) and the total system energy (3.9) can be rewritten in a similar manner, though we omit them for brevity.

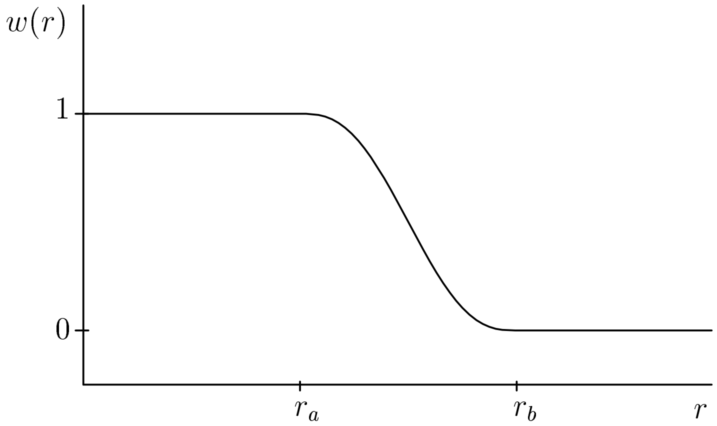

Figure 3.1: Potential weighting function.

3.2.1 Weighting Function

Ignoring distant neighbors, as described above, is sufficient under certain circumstances, but one must be careful. If at the neighborhood boundary, a given particle's contribution to the inter-particle force calculations result in a large enough discontinuity, then undesirable artifacts, such as instabilities in the numerical integration and visual artifacts, may occur. We present a solution to this problem now, and defer a detailed discussion to Chapter 7. Instead of ignoring particles outside of the neighborhood range, a better solution is to insure the inter-particle potentials, and hence forces, tend to zero at the neighborhood boundary.

We can enforce this condition by appropriately weighting the potential. To maintain the nature of the original potential, desirable properties of such a weighting function are that it be: monotonically decreasing from unity to zero; continuous and smooth; and continuous and smooth in the first and second derivatives. The smoothness and continuity conditions are important for well behaved numerical integration. To meet these conditions, we designed the following piecewise continuous function,

$$w(r) = \begin{cases} 1 & \text{if } r < r_a \\ g(s) & \text{if } r_a \le r \le r_b, \text{ letting } s = \frac{r - r_a}{r_b - r_a} \\ 0 & \text{if } r > r_b. \end{cases} , \qquad (3.12)$$

where r is the distance between two particles, and $0 < r_a < r_b$.

We implement g as a fifth degree interpolating polynomial

$$g(s) = -6s^5 + 15s^4 - 10s^3 + 1.0.$$

over the interval $[0, 1]$. We designed the polynomial such that the first and second derivatives of $g(s)$ are zero for $s = 0$ and $s = 1$. Figure 3.1 shows the graph of $w(r)$. We generally set r_a to be the standard spacing between particles and r_b to be equal to the neighborhood boundary range. In essence, the weighted potentials have compact support. Using such a weighting function we insure that summing weighted potentials over all particles is equivalent to summing weighted potentials over nearest neighbors, that is,

$$\sum_{j \neq i}^{N} w(r_{ij})\phi(r_{ij}) = \sum_{j \in \mathcal{N}_i}^{N} w(r_{ij})\phi(r_{ij})$$

where $r_{ij} = \|\mathbf{r}_{ij}\| = \|\mathbf{x}_j - \mathbf{x}_i\|$ is the distance between particles i and j.

Having redefined the potential with a weighting function, the corresponding force is also redefined. The new force is given by evaluating the gradient with respect to the position

$$\mathbf{f}_i = -\nabla_{\mathbf{x}_i} \left(w(r_{ij})\phi(r_{ij}) \right) = \hat{\mathbf{r}}_{ij} \left(w \frac{\mathrm{d}\phi(r_{ij})}{\mathrm{d}r} + \phi \frac{\mathrm{d}w(r_{ij})}{\mathrm{d}r} \right), \tag{3.13}$$

where $\hat{\mathbf{r}}_{ij}$ is the unit vector in the direction of $\mathbf{r}_{ij}$, that is

$$\hat{\mathbf{r}}_{ij} = \frac{\mathbf{r}_{ij}}{\|\mathbf{r}_{ij}\|}.$$

The derivation of (3.13) can be found in Appendix C.6.

3.3 Creating Deformable Volumes

Deformable solids inspired by finite element theory have been modeled using hexahedral assemblies of point masses, springs, and damping elements (Terzopoulos and Fleischer, 1988a). In such spring-mass models the springs are *structural elements* that hold the object together. As two particles connected by a spring are separated, the force exerted by the spring steadily increases, pulling the point masses back together. To model a material that fractures, one can "break" a spring when it is stretched beyond a threshold distance. In general, spring-mass systems are good for modeling solids with fixed structure exhibiting small deformations and fracturing. It is not an adequate representation for modeling materials exhibiting large geometric deformations or changes in genus. Instead of a spring potential energy that encourages particles to maintain a fixed structure, we would like a potential energy that allows groups of particles to be separated and joined back together in new and different configurations.

Our goal is to provide an alternative model that allows for large changes in geometry, topology, and genus. To do so, we use a inter-particle potential that is elastic for small deformations, yet allowing for the rearrangement of elements over large deformations. To allow for rearrangement, without manually redefining the connections, the function is defined over all particle pairs, instead of a fixed set of pairs as in mass-spring systems.

3.3.1 Lennard-Jones Potential

The Lennard-Jones potential energy function fulfills these criteria (Heyes, 1998). It creates long-range attractive forces and short range repulsive forces which encourage

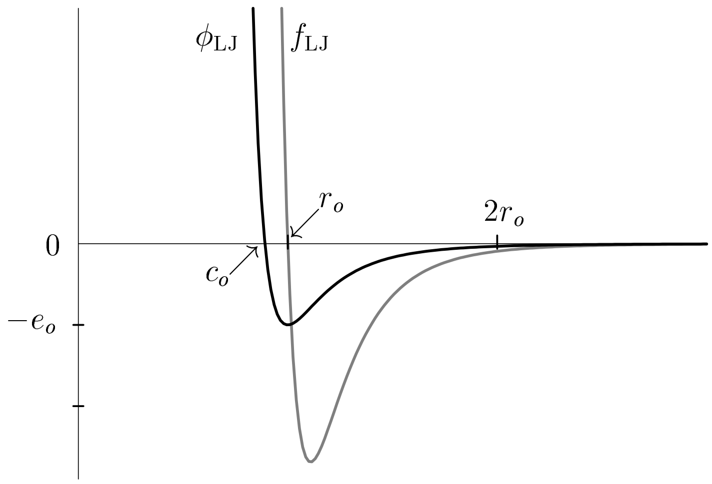

Figure 3.2: 12-6 Lennard-Jones function

The potential function for $n = 12$ and $m = 6$ is shown in black and the corresponding force shown in gray. The collision distance c_o, the equilibrium separation r_o, and the dissociation energy e_o are all labeled on the graph.

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particles to maintain equal spacing. It also allows particles to be rearranged relative to one another, and yet does not require the manual specification of inter-particle connections. The Lennard-Jones energy function is defined as a function of separation distance r between a pair of particles

$$\phi_{LJ}(r) = \frac{B}{r^n} - \frac{A}{r^m}. \tag{3.14}$$

In Figure 3.2, we show a Lennard-Jones function with $n = 12$ and $m = 6$, as typically used in the molecular dynamics literature. When two particles are in equilibrium, the potential energy between them is minimal and marked in Figure 3.2 at $-e_o$. The magnitude of this energy is known as the *dissociation energy* and is the energy required to completely separate two particles. The distance between two particles when in equilibrium is known as the *equilibrium separation distance*, r_o. The Lennard-Jones potential goes to zero at two points, at infinity and at a distance defined as the *collision distance*, c_o. These three quantities (e_o, r_o, c_o) are labeled in the figure.

An alternate formulation, called the Lennard-Jones bi-reciprocal function

$$\phi(r) = \frac{-e_o}{m-n}\left(m\left(\frac{r_o}{r}\right)^n - n\left(\frac{r_o}{r}\right)^m\right), \tag{3.15}$$

provides a convenient method of tailoring the potential function to a specific equilibrium separation and dissociation energy. This formulation is equivalent to (3.14) with

$$A = \frac{-e_o n {r_o}^m}{m-n} \qquad B = \frac{-e_o m {r_o}^n}{m-n}. \tag{3.16}$$

In this form it is easy to see that the collision distance is a function of the equilibrium separation r_o and exponents n and m. For the case of $n = 2m$ it is

$$c_o = r_o(2^{-1/m}). \tag{3.17}$$

Instead of using the typical molecular dynamic values of n, m, and r_o which correlate to the forces felt at the molecular level, we choose values which more closely mimic behavior found at the macroscopic level. At the macroscopic level a first order approximation of deformation can be modeled using a spring potential

$$\phi(r) = k(r - r_o)^2$$

between point masses (Terzopoulos and Fleischer, 1988a). In Figure 3.3 we compare the spring potential to the weighted and unweighted Lennard-Jones potential with $n = 4$ and $m = 2$. We define the constants r_a and r_b of the weighting function to be equal to r_o and $2r_o$ respectively. Close to the equilibrium separation, the Lennard-Jones function has a parabolic shaped potential energy well similar to that of the spring potential energy function.

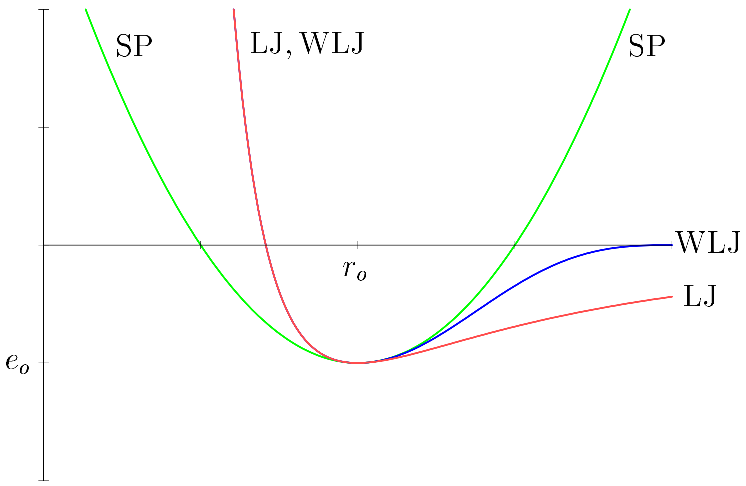
Similar to increasing the value k to increase stiffness in the spring model, we can increase the dissociation energy to increase stiffness in the Lennard-Jones particle model. Varying the exponents n and m varies the width of the potential well while keeping the minimum potential energy constant. Lower values result in wider potentials and more compressible materials, while higher values result in thinner potential wells and less compressible materials. A wide potential well will also result in flexible materials while a thin potential well will result in more rigid materials. Like large exponents, large dissociation energies will result in large forces and thus more rigid materials. The difference is incompressibility (the degree of volume preservation) and how "brittle" the material is. For equivalent forces, brittle materials (high exponents, low dissociation energy), require less energy to break the inter-particle bonds than non-brittle materials (low exponents, high dissociation energy).

3.3.2 Damping

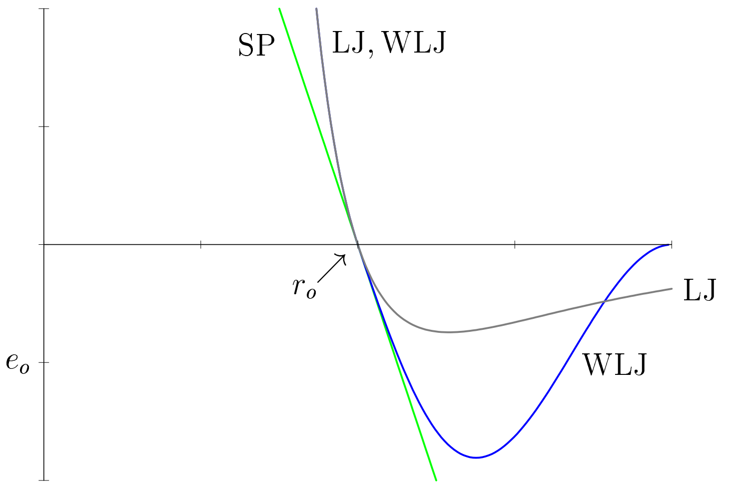
The second term of (3.1) is a velocity based damping force, $-\gamma_i \dot{\mathbf{x}}_i$. When $\gamma_i > 0$, this term accounts for a loss of kinetic energy, thereby allowing our energy minimizing system to eventually come to rest. It also provides a measure of inelasticity to collisions which is more typical of how we expect a synthetic material to behave. A viscous damping unit force function is given by

$$-\beta_1 \left(\dot{\mathbf{x}}_i - \dot{\mathbf{x}}_j\right), \tag{3.18}$$

where β_1 is the damping coefficient and $\dot{\mathbf{x}}_i$ and $\dot{\mathbf{x}}_j$ are particle velocities. This velocity based damping force differs from damping with respect to the world reference frame, $-\gamma_i \dot{\mathbf{x}}_i$, because it is independent of rigid body motion. Instead of decreasing the momentum of a single particle, it transfers momentum between neighboring particles.



(a) Potentials



(b) Forces

Figure 3.3: Spring and Lennard-Jones compared.

Potential energies and corresponding forces shown. Spring labeled SP. Weighted Lennard-Jones labeled WLJ. Lennard-Jones is labeled LJ. The spring parameters are $r_o = 1$ and $k = 8$. The Lennard-Jones parameters are $m = 2, n = 4, e_o = 1$, and $r_o = 1$. The weighting function parameters are $r_a = r_o$, and $r_b = 2r_o$.

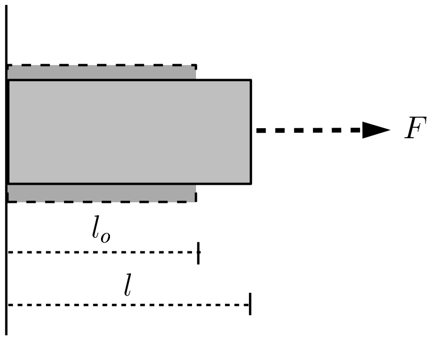


Figure 3.4: Longitudinal deformation

3.3.3 Rheology

Rheology is the science of the deformation and flow of matter. According to rheology, all materials can be modeled as falling between the two extremes of pure elastic (solid metal) and pure viscous (fluid) behavior (Houwink and de Decker, 1971). No real materials exhibit ideally elastic or ideally viscous behavior, though there are materials which come close. Some examples of elastic behavior are the steel spring and vulcanized rubber. Examples of viscous behavior include liquids such as water at room temperature, syrups, and molten glass. A few examples of mixed elastic and viscous behavior are wood, silk, PVC, and nylon.

Elasticity

A material that behaves according to ideal elastic behavior means that the material deforms under force and returns to its original form after release of the force. The resistance to deformation is described by the Young's modulus of elasticity E

$$F = E\epsilon \tag{3.19}$$

where F is the normal stress (force) and ϵ is the relative elongation

$$\epsilon = \frac{l - l_o}{l_o}$$

as shown in Figure 3.4.

The modulus of elasticity E is a physical constant only when the deformation is proportional to the stress. A linear approximation of Young's modulus can be computed using the Taylor series expansion. We compute it for small deformations near the equilibrium separation

$$F(r) = F(r_o) + (r - r_o)\frac{\mathrm{d}F(r_o)}{\mathrm{d}r} + O((r - r_o)^2).$$

Ignoring higher order terms we have

$$\begin{aligned} F(r) &= F(r_o) + \frac{\mathrm{d}F(r_o)}{\mathrm{d}r}(r - r_o) \\ &= -\frac{e_o nm}{r_o^2}(r - r_o). \end{aligned}$$

For this case, Young's modulus is

$$E = -\frac{e_o nm}{r_o}$$

and we have derived the equation for ideal linear elastic deformation, that is Hooke's law

$$F(r) = k(r - r_o),$$

where $k = E/r_o$.

Yield limit

Real materials have definite limits beyond which deformations are no longer elastic. For example, solid steel or glass wires do not yield elastically beyond 1% of their length. Nylon fibers can yield 20% of their length at room temperature. An unusual example, vulcanized rubber can yield 500% of its length (Houwink and de Decker, 1971). The limit after which materials are no longer elastic is known as the yield limit. After this limit other mechanisms take over, such as plasticity or flow behavior. Under some conditions there is no mechanism that takes over, in which case the material breaks. The pieces of such a broken object can be fitted back together to reconstruct the original shape. The breaking of ceramic coffee mugs or everyday drinking glasses are examples.

The tensile strength of a material is a measure of the maximum force value exhibited by a material in response to a stress. In the case of the Lennard-Jones function, the force increases as the particles separate until the maximum force at the separation value $r = r_t$ is reached and then it decreases until force goes to zero at infinity or the distance r_b, in the case of a weighted potential. The maximum force is when

$$\frac{d^2\phi_{\mathrm{LJ}}(r)}{dr^2} = 0,$$

which is equivalent to the condition

$$r^{n-m} = r_o^{n-m}\frac{n+1}{m+1}.$$

The magnitude of the tensile strength is the depth of the well in the force function as shown in Figure 3.3 (b). Under forces greater than the tensile strength, the bond will fracture.

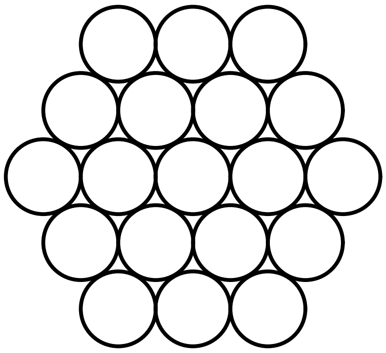


Figure 3.5: Hexagonal packing.

Plasticity

Plastic or visco-elastic behavior is the response of materials that combine both elastic and viscous responses to stress and can be characterized by the additivity of the stress. The Voight model (Jaeger, 1969) of plasticity states

$$F = E\epsilon + \eta \frac{\mathrm{d}\epsilon}{\mathrm{d}t}$$

where F is the force and η is a viscous term. Our model is similar to the Voight model with the elastic portion represented by the Lennard-Jones force and the viscous portion represented by the viscous damping unit equation (3.18).

For stresses less than the maximum force value, the particles will separate by less than the yield separation value of $r_t - r_o$, and will return to the equilibrium separation when the force is removed. For stresses greater than the tensile strength value, particle bonds may be broken and permanent deformations may occur as particles are rearranged relative to one another.

3.4 Packings

Since the Lennard-Jones function is defined in terms of the Euclidean distance from a particle, it is a *spatially symmetric* potential energy function. Given two particles i and j the set of minimum energy states (positions) for particle j relative to i is the locus of points of a sphere centered at the position of particle i. When external forces are insignificant, particles arrange into closely packed structures to minimize their total energy.

For spherically symmetrical potential energy functions in 2-D, such as the Lennard-Jones potential, the particles arrange into hexagonal orderings as shown in Figure 3.5. In 3-D the particles arrange into hexagonal ordered 2-D layers, making Lennard-Jones good for modeling volumes of material.

3.4.1 Density

The dense packing of particles lets us ask "How many particles will it take to fill a given volume?" We have found this to be a common question. The answer is also used in computing the efficiency of neighboring finding techniques given in Section 7.2. We can approximate the answer by considering the related question "How many solid equal size spheres will it take to fill a given volume?" The *volume packing factor* is defined as the ratio of the unit-sphere to the unit-enclosure (Gasson, 1983). For a given packing, this is equivalent to the ratio of the volume of a sphere to the associated 3D Voronoi region, where the Voronoi diagram is computed over the center points of the spheres. A hexagonal close packing of spheres has a volume packing factor of P_v

$$P_v = \frac{\pi}{3\sqrt{2}} \approx 0.74. \tag{3.20}$$

In comparison, a cubic packing of spheres, where each sphere is positioned at the corner of a cubic lattice, has a packing factor of $\pi/6 \approx 0.52$ which is significantly less than the hexagonal packing factor. Given the packing factor, the expected number of spheres n, in a volume V, is given by

$$n = P_v \frac{V}{V_s},$$

where $V_s = \frac{4}{3}\pi r^3$ is the volume of a sphere with radius r. For hexagonal packing this reduces to

$$n = \frac{V}{4\sqrt{2}r^3}.$$

In accordance to the Lennard-Jones force, at equilibrium two particles will be separated by the equilibrium separation r_o. Thus we approximate the volume of a particle

$$V_p = \frac{V_s}{P_v} = \frac{{r_o}^3}{\sqrt{2}}, \tag{3.21}$$

as the volume of sphere, with a radius of one half the equilibrium separation, divided by the hexagonal packing factor. When the neighborhood range includes only particles of a distance r_o this accurately represents the effective particle volume. However when the neighborhood range includes more distant particles, the particles will be packed more closely. The amount of compression depends on the strength of the attractive forces and any external pressure on the system.

3.5 External Forces

Our model of volumetric shape becomes more interesting when we put it in an environment with external forces and obstacles. In this section we describe two such external forces.

3.5.1 Gravity

We add the force of gravity, $\mathbf{f} = \mathbf{g}m_i$, where $\mathbf{g}$ is a gravitational acceleration in a given direction and m_i is the mass of particle i, to our simulated world so that we can drop objects and pour fluids.

3.5.2 Collisions

We introduce obstacles into our environment and create collision forces so that particles will not penetrate these objects. A repulsive force is defined between each particle and object surface similar to the repulsive force between particles. The repulsive force is limited to a short range so that particles are only repelled when they are very close to the surface. The force is based on inverse powers of the distance between the object and particle. As a particle and object collide, the particle will slow down due to the repulsive force and gain potential energy relative to the obstacle.

For an object k and particle i separated by a distance d_{ki}, we define the collision potential energy function as

$$e_c\left(d_{ki}\right) = \begin{cases} \alpha/d_{ki}^{\eta} + \beta d_{ki} + \gamma & \text{when } d_{ki} \leq d_o \\ 0 & \text{otherwise,} \end{cases} \tag{3.22}$$

and the resulting force as

$$f_c\left(d_{ki}\right) = \begin{cases} \frac{n\alpha}{d_{ki}^{\eta+1}} - \beta & \text{when } d_{ki} \leq d_o \\ 0 & \text{otherwise.} \end{cases} \tag{3.23}$$

The distance d_o is the distance from the obstacle surface at which the particle gains potential energy relative to the obstacle. By constraining $e_c\left(d_o\right) = 0$ and $f_c\left(d_o\right) = 0$, the potential energy and force functions are continuous for $d_{ki} > 0$. For a constant value of η, the user need only specify the distance $d_o > 0$ and the scaling factor $\alpha > 0$, and the remaining constants are uniquely determined. This works for all obstacles given there is a function for computing the shortest distance between a point and the surface.

3.6 Examples

3.6.1 Deformations

Figures 3.6 and 3.7 shows an object modeled using volume particles. The parameters of the Lennard-Jones potential are varied to make one version of the model rigid and the other flexible. In both figures the bonds between particles are strong enough so that the objects behave as solids, maintaining their structure under the influence of external forces, in this case gravity and collision forces. In this example we varied the exponents of the Lennard-Jones potential, thus increasing the maximum binding force, without increasing the dissociation energy. That is it “sharpens” the potential

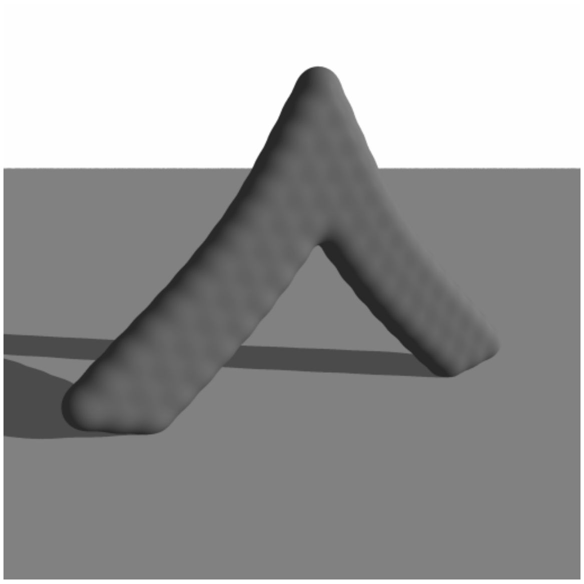


Figure 3.6: Flexible solid

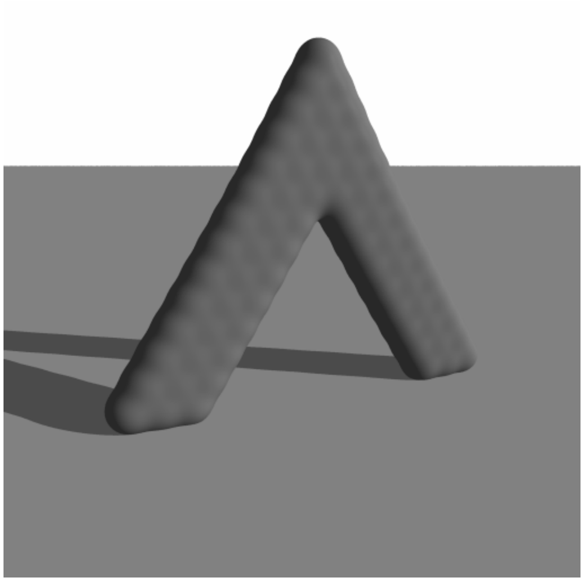


Figure 3.7: Rigid solid

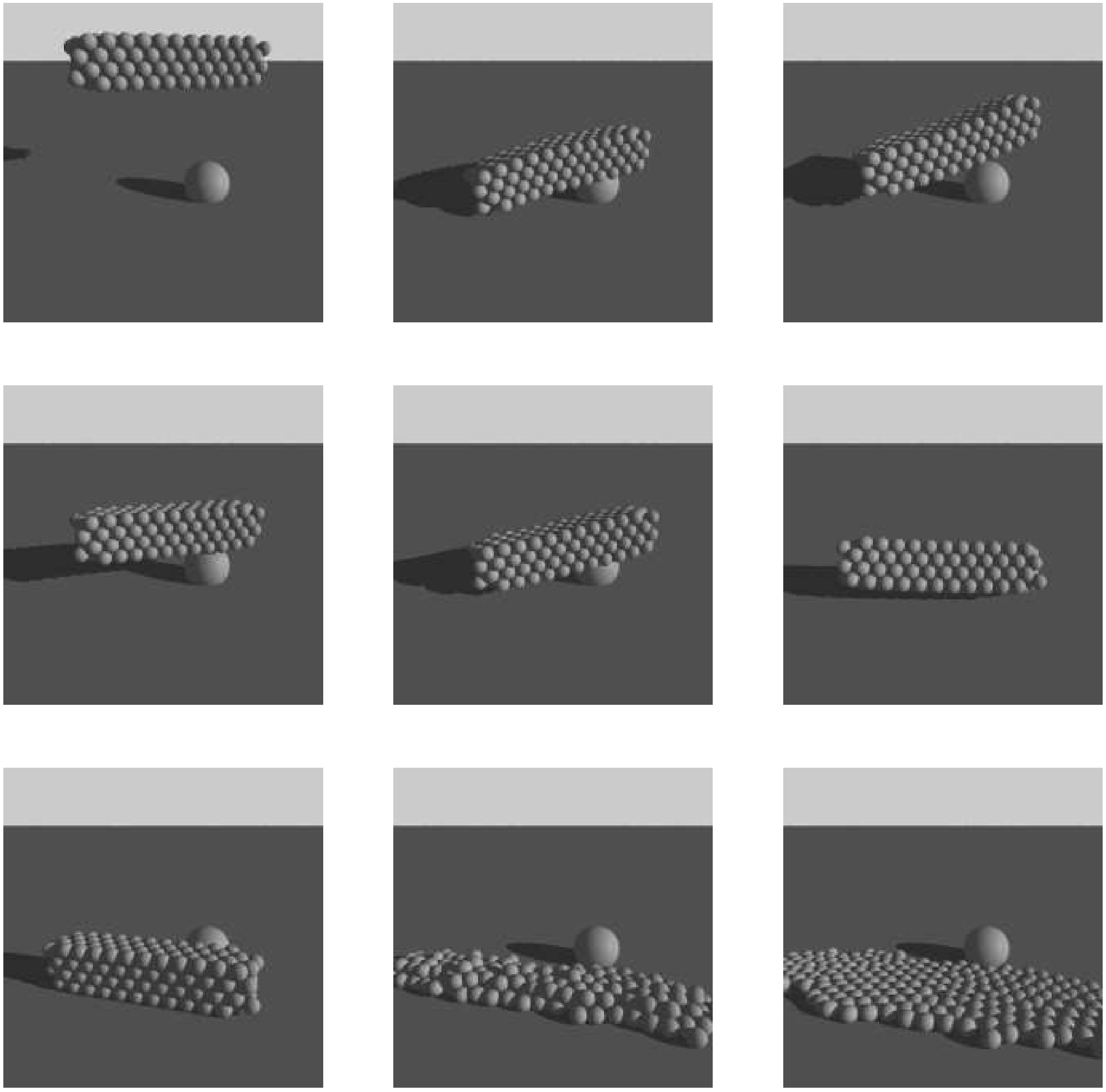


Figure 3.8: Beam colliding

A solid beam falling, colliding with obstacles. The frames were taken from an animation at the following times: $t = 0, 3, 4, 5, 9, 12, 25, 70, 83$. In the second frame the beam collides with the sphere, and floor (not shown), causing it to bounce back into the air (frames 3 and 4). Note the slight flexing of the beam. In frames 6 and 7 it rolls forward, further displaying rigid body motion. In the last two frames the magnitude of the dissociation energy is reduced (see Chapter 6) and the initial structure of the object is lost.

well without "deepening" it. This creates rigid materials, but also ones that are "brittle".

Another approach to creating rigid solids is to increase the magnitude of the dissociation energy. This also increases the maximum binding force, but creates materials that are more durable and less likely to break under sudden external forces. The bouncing beam in Figure 3.8 is an example using this approach. In this example, a solid beam falls colliding with obstacles, exhibiting rigid body motion with slight deformation. After striking a sphere and the floor plane the beam bounces back into the air. After the second collision with the sphere it rolls forward coming to rest on the ground plane. In the last few frames the dissociation energy is reduced and the solid looses its initial structure much like the varying structure found in fluids.

For Figures 3.6 and 3.7 the Lennard-Jones parameters are equilibrium separation $r_o = 1.192$, dissociation energy of $e_o = 200$, and the exponents values $(n = 4, m = 2)$ and $(n = 8, m = 6)$ for the flexible and rigid solids respectively. The remaining parameters are as follows. The particles have mass $m_i = 1$. The gravitational constant is $g = 1.5$. The collision force constants are $\beta = 25$, $d_o = 1$, and $\eta = 4$. The velocity based damping constant is $\gamma = 1.5$ and the viscous based damping constant is $\beta_1 = 0$. Each solid is composed of 69 particles. The numerical integration was performed according to the Euler method (Chapter 7) and rendered as iso-surfaces (Chapter 7).

We have begun our development of a flexible shape model which allows the user to create objects of arbitrary topology. The goal of the model is to provide a simple yet powerful method of modeling objects whose local shape and geometry change rapidly over time. An important criterion, implicit in this goal, is the ability to make large changes at the global level, such as changes in topology. It is also important for the user to be able to control the physical characteristics of the model, such as the relative amount of deformation due to a given force. We have described a particle based model suitable for representing volumes and in the next chapter we extend this model to surfaces.

The approach taken to satisfy these requirements is to represent the object as a collection of primitive elements, whose relative positions dictate the shape and geometry of the object. That is, different shapes and geometries will have different arrangements of the volume elements. Similar to the use of pixels (picture elements) to portray an unlimited range of two dimensional images, the use of simple elements can be used to describe an unlimited number of shapes.

The model, based on dynamically coupled particle systems, describes changes in geometry and the movement of element volumes, as a consequence of external forces and internal potential energies. By varying the internal potential energies we can model a variety of physical properties ranging from stiff to fluid like behavior. Timing results can be found in Section 7.3.5.