CSC 236
St George Campus

November 4, 2004
University of Toronto

Solutions to Assignment #3

Answer to Question 1.

$\rightarrow$ is not associative. For example, any truth assignment that falsifies P and R (independent of what it does to Q), satisfies $P \rightarrow (Q \rightarrow R)$ but falsifies $(P \rightarrow Q) \rightarrow R$.

$\leftrightarrow$ and $\oplus$ are both associative as shown by the following truth tables.

P	Q	R	$P \leftrightarrow (Q \leftrightarrow R)$	$(P \leftrightarrow Q) \leftrightarrow R$	$P \oplus (Q \oplus R)$	$(P \oplus Q) \oplus R$
0	0	0	0	0	0	0
0	0	1	1	1	1	1
0	1	0	1	1	1	1
0	1	1	0	0	0	0
1	0	0	1	1	1	1
1	0	1	0	0	0	0
1	1	0	0	0	0	0
1	1	1	1	1	1	1

(Note that all four formulas have exactly the same truth table: A truth assignment satisfies each formula if and only if an odd number of P, Q and R are satisfied by the truth assignment.)

"|" (***nand***) is not associative. For example, a truth assignment that satisfies P, and falsifies Q and R, satisfies $(P \mid Q) \mid R$ but falsifies $P \mid (Q \mid R)$.

Answer to Question 2.

a. $((x \vee y) \rightarrow z) \;\vee\; (z \rightarrow (x \vee y))$ is a tautology.

Proof: The only way for $((x \vee y) \rightarrow z)$ to be false is for $x \vee y$ to be true and z to be false, in which case $(z \rightarrow (x \vee y))$ is true.

b. $(x \rightarrow y) \;\vee\; (x \rightarrow \neg\, y)$ is a tautology.
Proof:

$$\begin{aligned} & (x \rightarrow y) \vee (x \rightarrow \neg\, y) \\ \text{LEQV } & (\neg x \vee y) \vee (\neg x \vee \neg\, y) \\ \text{LEQV } & \neg x \vee \neg x \vee \neg\, y \vee y \end{aligned}$$

which is a tautology because of the $\neg\, y \vee y$.

c. $((x \rightarrow y) \wedge (y \rightarrow z)) \;\leftrightarrow\; (x \rightarrow z)$ is a contingency. If x, y, and z are all false, then all of the $\rightarrow$ operators yield true, so both sides of the "$\leftrightarrow$" are true. If x and z are both true but y is false, then $x \rightarrow y$ is false, so the left side of the "$\leftrightarrow$" is false, but the right side of the "$\leftrightarrow$" is true.

d. $((x \rightarrow y) \wedge (x \rightarrow z)) \;\rightarrow\; (x \rightarrow (y \wedge z))$ is a tautology.

Proof:

$$\begin{aligned}
& (x \to y) \wedge (x \to z) \\
\text{LEQV } & (\neg x \vee y) \wedge (\neg x \vee z) \\
\text{LEQV } & \neg x \vee (y \wedge z) \\
\text{LEQV } & x \to (y \wedge z)
\end{aligned}$$

e. $((x \to y) \vee (x \to z)) \ \to \ (x \to (y \vee z))$ is a tautology.
Proof:

$$\begin{aligned}
& (x \to y) \vee (x \to z) \\
\text{LEQV } & (\neg x \vee y) \vee (\neg x \vee z) \\
\text{LEQV } & \neg x \vee (y \vee z) \\
\text{LEQV } & x \to (y \vee z)
\end{aligned}$$

f. $((x \wedge y) \to (x \wedge z)) \ \to \ (x \to (y \to z))$ is a tautology.
Proof:

$$\begin{aligned}
& (x \wedge y) \to (x \wedge z) \\
\text{LEQV } & \neg(x \wedge y) \vee (x \wedge z) \\
\text{LEQV } & \neg x \vee \neg y \vee (x \wedge z) \\
\text{LEQV } & \neg y \vee \neg x \vee (x \wedge z) \\
\text{LEQV } & \neg y \vee (\neg x \vee x) \wedge (\neg x \vee z) \\
\text{LEQV } & \neg y \vee (\neg x \vee z) \\
\text{LEQV } & \neg x \vee (\neg y \vee z) \\
\text{LEQV } & \neg x \vee (y \to z) \\
\text{LEQV } & x \to (y \to z)
\end{aligned}$$

Answer to Question 3.

a. Lemma: If P is the only propositional variable and P has the value *true*, then all propositional formulas involving no operators other than → also evaluate to *true*.
Proof:

Base case: The only formula with no operators at all is P, which has the value *true*.

Inductive step: Suppose that F and G are propositional formulas involving no propositional variables other than P and no operators other than →, and that we know that when P has the value *true*, F and G evaluate to *true*.

Then $F \to G$ evaluates to $true \to true$ which is *true*.

Thus the lemma has been proved. Now consider the formula $\neg P$. This evaluates to $false$. Since all propositional formulas involving no operators other than → evaluate to *true*, we know that $\neg P$ cannot be equivalent to any of them.

Therefore the set $\{\to\}$ is not a complete set of boolean connectives.

b. Here is a mechanical transformation to convert any propositional calculus formula to a logically equivalent formula involving only the $\rightarrow$ operator, the propositional variables, and the constant $false$. (The resulting formula would not be accepted as a propositional formula in the traditional sense.)

First, convert the formula to DNF using the technique involving reading the formula from the truth table.

Next, convert the formula to use only the $\neg$ and $\vee$ operators by applying deMorgan's law to each of the disjuncts (which are themselves conjunctions of propositions and negated propositions), as we did to demonstrate that $\{\neg, \vee\}$ is a complete set of connectives.

Next, replace all $\neg$ operators as follows: any well-formed formula of the form $\neg F$ is transformed to $(F \rightarrow false)$. For example, $\neg(P \vee \neg Q)$ is transformed to $((P \vee (Q \rightarrow false)) \rightarrow false)$. This transformation produces a logically equivalent formula because $(P \rightarrow false)$ LEQV $\neg P$ (this is most easily verified by a truth table, which will only be two lines long). At this point the formula contains only $\vee$ and $\rightarrow$ connectives.

Finally, replace all subformulas containing a $\vee$ with formulas involving only $\rightarrow$, by replacing each well-formed subformula of the form $(F \vee G)$ with $((F \rightarrow false) \rightarrow G)$. This transformation again produces a logically equivalent formula because $((F \rightarrow false) \rightarrow G)$ LEQV $(\neg F \rightarrow G)$ LEQV $(\neg\neg F \vee G)$ LEQV $(F \vee G)$. At this point the formula contains only $\rightarrow$ connectives.

Therefore for every well-formed propositional calculus formula there is a logically equivalent formula containing only the connective $\rightarrow$, the propositional variables, and the constant $false$.

Answer to Question 4.

Note that $\leftrightarrow$ can be read as "equals", and that $\oplus$ can be read as "not equal to".

I express my reasoning below in terms of a transformation from the number x to the number $x + 1$.

First of all, the x_0 bit indicates whether or not the number is odd; specifically, it is equal to x_τ **mod** 2. Adding one thus changes this bit; if $y_\tau = x_\tau + 1$ then $y_0 \neq x_0$, i.e. $y_0 \oplus x_0$ is true.

The x_1 bit does not always flip. It will flip iff the adding of 1 to x_0 produces a "carry" to the left. This is true iff $x_0 = 1$. Otherwise, if $x_0 = 0$, this bit remains the same. Thus $y_1 = x_1$ iff $x_0 = 0$ (i.e. if $x_0 = 0$, y_1 is *not* equal to x_1, meaning that this bit flips; and furthermore, if $x_0 = 1$, y_1 *is* equal to x_1; all this is stated by this iff sentence). As a propositional calculus formula, this is the statement that $(y_1 \leftrightarrow x_1) \oplus x_0$.

Similarly we can write propositional formulas which state whether or not x_2 and x_3 flip. They flip iff all bits to the "right" are 1. That is, x_2 flips iff $x_1 \wedge x_0$, and x_3 flips iff $x_2 \wedge x_1 \wedge x_0$. So we have that $(y_2 \leftrightarrow x_2) \oplus (x_1 \wedge x_0)$ and $(y_3 \leftrightarrow x_3) \oplus (x_2 \wedge x_1 \wedge x_0)$.

There is no x_4 to consider whether or not it flips. y_4 will be 1 if x_τ is the binary number 1111, and will be 0 if x_τ is anything else. That is to say, $y_4 \leftrightarrow (x_3 \wedge x_2 \wedge x_1 \wedge x_0)$.

The formula is the conjunction of all of these. That is, *all* of these statements about the y_is must be true for it to be the case that $y_\tau = x_\tau + 1$.

So the formula is:

$$(y_0 \oplus x_0) \wedge ((y_1 \leftrightarrow x_1) \oplus x_0) \wedge ((y_2 \leftrightarrow x_2) \oplus (x_1 \wedge x_0)) \wedge ((y_3 \leftrightarrow x_3) \oplus (x_2 \wedge x_1 \wedge x_0)) \wedge (y_4 \leftrightarrow (x_3 \wedge x_2 \wedge x_1 \wedge x_0))$$