Answer to Question 5. Assume the Precondition holds.

Loop Invariant Lemma: For each $k \in \mathbb{N}$, if the loop is executed at least k times then (a) $t_k = b^{i_k}$ and (c) $b^{i_k} \leq x$.

Before proving the LIL, we show that it implies partial correctness. Assume the program terminates. Thus, the loop terminates after some number of iterations, say r. By the loop exit condition, $b \cdot t_r > x$. By the LIL, $t_r = b^{i_r}$ and $b^{i_r} \leq x$. Thus, $b^{i_r} \leq x < b^{i_r+1}$. The program returns the value of i_r. Thus the program returns the natural number ℓ with the property that $b^\ell \leq x < b^{\ell+1}$, i.e., it retuns the integer logarithm of x base b.

We now prove the LIL. Consider the predicate

$$P(n): \qquad \text{If the loop is executed at least } n \text{ times then (a) } t_n = b^n \text{ and (b) } b^n \leq x.$$

We will use simple induction to prove that $P(n)$ holds for every $n \in \mathbb{N}$.

BASIS: $n = 0$. By the program, $i_0 = 0$ and $t_0 = 1 = b^{i_0}$. By the precondition, $b^{i_0} = 1 \leq x$. Thus, $P(0)$ holds.

INDUCTION STEP: Let k be an arbitrary natural number, and assume that $P(k)$ is true. We must prove that $P(k+1)$ is also true.

Assume that the loop is executed at least $k+1$ times (otherwise $P(k+1)$ is trivially true). Then,

$$\begin{aligned}
t_{k+1} &= t_k \cdot b && \text{[by the program]}\\
&= b^{i_k} \cdot b && \text{[by the induction hypothesis]}\\
&= b^{i_k+1} &&\\
&= b^{i_{k+1}} && \text{[by the program]}
\end{aligned}$$

$$\begin{aligned}
x &\geq b \cdot t_k && \text{[because the loop is executed } \geq k+1 \text{ times]}\\
&= b \cdot b^{i_k} && \text{[by the induction hypothesis]}\\
&= b^{i_k+1} &&\\
&= b^{i_{k+1}} && \text{[by the program]}
\end{aligned}$$

Thus, $P(k+1)$ is true.

It remains to prove termination. For this it suffices to identify a quantity that is always a non-negative integer and (strictly) decreases in each iteration. For our program, such a quantity is $x - t$. More precisely, we claim that for every $k \in \mathbb{N}$, if the loop is executed at least k times then (i) $x - t_k$ is a natural number, and (ii) if $k > 0$ then $x - t_k < x - t_{k-1}$.

Part (i) follows immediately from the fact that $x \in \mathbb{N}$ (by the precondition), $t_k \in \mathbb{N}$ (by the precondition and the LIL), and $t_k \leq x$ (by the LIL).

Part (ii) follows because $t_k = t_{k-1} \cdot b$ and $b > 1$ (by the precondition).