

Solutions for Assignment #1

**Answer to Question 1.**

Base case:

$$\frac{1}{\sqrt{1}} = 1$$

Now assume that

$$\sum_{i=1}^k \frac{1}{\sqrt{i}} \leq 2\sqrt{k} - 1$$

for some  $k \geq 1$ .

We need to prove that

$$2\sqrt{k} - 1 + \frac{1}{\sqrt{k+1}} \leq 2\sqrt{k+1} - 1$$

which is equivalent to

$$2\sqrt{k} + \frac{1}{\sqrt{k+1}} \leq 2\sqrt{k+1}$$

Proof:

Since  $k > 0$ ,

$$\sqrt{4k^2 + 4k} \leq \sqrt{4k^2 + 4k + 1}$$

and since  $4k^2 + 4k + 1 = (2k + 1)^2$ ,

$$2\sqrt{k^2 + k} \leq 2k + 1$$

Factor left side and add 1 to both:

$$2\sqrt{k}\sqrt{k+1} + 1 \leq 2(k+1)$$

Divide by  $\sqrt{k+1}$ :

$$2\sqrt{k} + \frac{1}{\sqrt{k+1}} \leq 2\sqrt{k+1}$$

■

**Answer to Question 2.** Consider the predicate

$$P(n) : \quad 7^n - 2^n \text{ is a multiple of 5.}$$

We will use simple induction to prove that  $P(n)$  holds for every  $n \in \mathbb{N}$ .

**BASIS:**  $n = 0$ . In this case  $7^n - 2^n = 1 - 1 = 0$ . Since 0 is a multiple of 5,  $P(0)$  holds.

**INDUCTION STEP:** Let  $i$  be an arbitrary integer such that  $i \geq 1$ . Assume that  $P(i)$  is true; let  $q \in \mathbb{N}$  be such that  $7^i - 2^i = 5q$ . We must prove that  $P(i+1)$  is also true; that is, we must prove that for some  $q' \in \mathbb{N}$ ,  $7^{i+1} - 2^{i+1} = 5q'$ . We have

$$\begin{aligned} 7^{i+1} - 2^{i+1} &= 7 \cdot 7^i - 2 \cdot 2^i \\ &= 7 \cdot 7^i - 2 \cdot (7^i - 5q) && \text{[by the induction hypothesis]} \\ &= 5 \cdot 7^i + 5 \cdot 2q \\ &= 5 \cdot (7^i + 2q) \end{aligned}$$

Thus, taking  $q' = 7^i + 2q$ , we have shown that there is some  $q' \in \mathbb{N}$  so that  $7^{i+1} - 2^{i+1} = 5q'$ . Therefore,  $P(i + 1)$  holds.

**Answer to Question 3.**

a. Consider the predicate

$$P(h) : \quad \text{every height-balanced tree of height } h \text{ has at least } 1.6^h \text{ nodes.}$$

We will use complete induction to prove that  $P(h)$  holds for every  $h \in \mathbb{N}$ .

Let  $i$  be an arbitrary natural number. Assume that  $P(j)$  holds for every integer  $j$  such that  $0 \leq j < i$ . We will prove that  $P(i)$  holds as well.

CASE 1.  $i = 0$ . There is only one height-balanced tree of height 0, namely one that has a root with no children. Thus the only height-balanced tree of height 0 has 1 node. Since  $1 \geq 1.6^0$ ,  $P(0)$  holds.

CASE 2.  $i = 1$ . There are exactly three height-balanced trees of height 1: one has a root with a left child that is a leaf and no right child, one has a root with a right child that is a leaf and no left child, and one has a root with a left and a right child, both of which are leaves. Therefore every height-balanced tree of height 1 has at least two nodes, and  $2 \geq 1.6^1$ , so  $P(1)$  holds.

CASE 3.  $i \geq 2$ . Let  $T$  be any height-balanced tree of height  $i$ . Consider the two subtrees of  $T$  (i.e., the two subtrees rooted at the children of  $T$ 's root). Since  $T$ 's height is  $i$ , one of its subtrees, say  $T_1$ , must have height *exactly*  $i - 1$ ; while the other, say  $T_2$ , must have height *at most*  $i - 1$ . But because  $T$  is height-balanced, the height of  $T_2$  must be at least  $i - 2$ . Furthermore, both  $T_1$  and  $T_2$  must be height-balanced (otherwise  $T$  would not be height-balanced). Therefore,  $T_1$  is a height-balanced tree of height  $i - 1$ , and  $T_2$  is a height-balanced tree of height  $i - 2$  or  $i - 1$ . Since  $i \geq 2$ ,  $0 \leq i - 2 < i - 1 < i$ , and so by the induction hypothesis  $P(i - 2)$  and  $P(i - 1)$  both hold. Therefore, the number of nodes  $n_1$  of  $T_1$  is at least  $1.6^{i-1}$ , and the number of nodes  $n_2$  of  $T_2$  is at least  $1.6^{i-2}$ .

Let  $n$  be the number of nodes of  $T$ . Since the nodes of  $T$  are its root and the nodes of  $T_1$  and  $T_2$ , we have:

$$\begin{aligned} n &= n_1 + n_2 + 1 \\ &> 1.6^{i-1} + 1.6^{i-2} \\ &= 1.6^{i-2} \cdot 2.6 \\ &> 1.6^{i-2} \cdot 1.6^2 \\ &= 1.6^i. \end{aligned}$$

Therefore  $T$  has at least  $1.6^i$  nodes, and so  $P(i)$  holds.

b. It is not true that any binary tree of height  $h$  has at least  $1.6^h$  nodes. For example, consider the following binary tree with four nodes  $u_1, u_2, u_3$  and  $u_4$ . Node  $u_1$  is the root, and  $u_i$  is the left child of  $u_{i-1}$ , for all  $i$  such that  $1 < i \leq 4$ . No node has a right child. This tree has four nodes and height 3. Since  $1.6^3 = 4.096$ , it is not true that every binary tree of height 3 has at least  $1.6^3$  nodes.

**Answer to Question 4.** Using the definition of  $f$  we compute the following table:

$n$	$f(n)$	$n^2/2$
0	10	0
1	6	$\frac{1}{2}$
2	3	2
3	13	$4\frac{1}{2}$
4	10	8
5	8	$12\frac{1}{2}$
6	19	18
7	17	$24\frac{1}{2}$
8	16	32
9	28	$40\frac{1}{2}$

We claim that  $d = 7$  is the smallest number with the required property. To prove this, it suffices to prove that (a) for every integer  $n \geq 7$ ,  $f(n) \leq n^2/2$ , and (b)  $f(6) > 6^2/2$ .

Part (b) is immediate from the above table.

To prove (a), let  $P(n)$  be the predicate “ $f(n) \leq n^2/2$ ”. We use complete induction to prove that  $P(n)$  is true for every integer  $n \geq 7$ . Let  $i$  be an arbitrary integer greater than or equal to 7. Assume that  $P(j)$  is true for every integer  $j$  such that  $7 \leq j < i$ . We must prove that  $P(i)$  is also true.

CASE 1.  $i \in \{7, 8, 9\}$ . The fact that  $P(i)$  holds in this case follows immediately from the above table.

CASE 2.  $i \geq 10$ . Thus  $7 \leq i - 3 < i$ , and so  $P(i - 3)$  holds by the induction hypothesis. We have

$$\begin{aligned}
 f(i) &= f(i - 3) + i && \text{[by def. of } f, \text{ since } i \geq 3\text{]} \\
 &\leq (i - 3)^2/2 + i && \text{[by the induction hypothesis]} \\
 &= i^2/2 - (4i - 9)/2 \\
 &\leq i^2/2 && \text{[since } i \geq 10 \text{ and so } 4i - 9 \geq 0\text{]}
 \end{aligned}$$

Thus  $P(i)$  holds in this case.

**Answer to Question 5.** Let  $S(n)$  be the statement “if there are  $n$  cities so that we can transport goods between them using only NUT trucks, then NUT must have at least  $n - 1$  trucks”. We use complete induction to prove that  $S(n)$  is true for every positive integer  $n$ .

Let  $k$  be an arbitrary positive integer. Suppose that  $S(j)$  is true for every integer  $j$  such that  $1 \leq j < k$ . We will prove that  $S(k)$  is also true.

CASE 1.  $k = 1$ . If there is only one city, then there are no two distinct cities to transport goods between, and NUT can get away with zero trucks in this case. So, in this case NUT must have at least  $0 = 1 - 1$  trucks, and  $S(1)$  holds.

CASE 2.  $k \geq 2$ . Consider  $k$  cities connected by truck routes as described in the problem. As mentioned in the hint, if we remove one of the cities, say  $c$ , and all the trucks that transport goods between  $c$  and other cities, the remaining  $k - 1$  cities can be grouped into a collection of  $\ell \geq 1$  nonempty sets  $C_1, C_2, \dots, C_\ell$ , so that we can transport goods between cities within the same set, but we cannot transport goods between cities in different sets. Let  $k_i$  be the number of cities in  $C_i$ , for all  $1 \leq i \leq \ell$ ; in other words,  $k_i = |C_i|$ . Clearly  $1 \leq k_i < k$ , since each  $C_i$  is nonempty, and does not contain at least one of the  $k$  cities, namely  $c$ . Therefore, by the induction hypothesis,  $S(k_i)$  holds; that is, the induction hypothesis applies to each  $C_i$ .

Consider any  $i$ ,  $1 \leq i \leq \ell$ . By the induction hypothesis, NUT has at least  $k_i - 1$  trucks transporting goods between cities in  $C_i$ . Furthermore, there must be at least one truck that transports goods between  $c$ , the city that was removed, and some city in  $C_i$ . (We can show this by contradiction. Suppose the contrary: that is, there is some  $i$  such that no NUT truck transports goods between  $c$  and some city in  $C_i$ . Let  $y$  be any city in  $C_i$ . By assumption, there is a way of transporting goods from  $x$  to  $y$ , and since there is no truck that does this directly, the transportation must be done via intermediate cities, some of which do not belong to  $C_i$ . Thus there is a way of transporting goods from a city in  $C_j$  (for some  $j \neq i$ ) to  $y$ , a city in  $C_i$ , without going through  $x$ . This contradicts the definition of the sets  $C_1, C_2, \dots, C_\ell$  into which the  $k - 1$  cities other than  $x$  were grouped.)

In summary, we have that for each set of cities  $C_i$ , NUT must have at least  $k_i - 1$  trucks to transport goods between the cities in that set, and one more truck to transport goods between  $x$  and some city in  $C_i$ , for a total of  $k_i$  trucks for each  $1 \leq i \leq \ell$ . We therefore conclude that NUT must have at least  $\sum_{i=1}^{\ell} k_i$  trucks. The total number of cities is  $k = 1 + |C_1| + |C_2| + \dots + |C_\ell| = 1 + \sum_{i=1}^{\ell} k_i$  (the extra 1 is to account for the city  $x$ , which is the only city that is not in one of the  $C_i$ s). Therefore, when there are  $k$  cities NUT must have at least  $k - 1$  trucks. So,  $S(k)$  holds.